# THE SPECTRUM OF THE LAPLACIAN: A GEOMETRIC APPROACH

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**Preamble**: These notes correspond to a 4-hour lecture and one exercise session given in Montréal in June 2015 during the summer school *Geometric and Computational Spectral Theory*. The goal was to introduce the subject, that is to present various aspects of the spectrum of the Laplacian on a compact Riemannian manifold from a geometric viewpoint, and also to prepare the audience for some of the following lectures.

In the first part, I recall without proof some classical facts, I try to illustrate the theory with examples and I discuss open questions. The aim is to give an intuition to the students discovering the subject, without being too formal. The exercise session was very useful, revealing that things which are obvious for the people of the field are far from being simple for those looking at the question for the first time. This is the reason why I decided to present in these notes really simple examples with a lot of details.

In the second part, I focus mainly on a specific problem: the estimates of the spectrum in the conformal class of a given Riemannian metric. This is the opportunity to present recent developments on the subject and to explain some very interesting geometrical methods in order to get upper bounds for the spectrum.

As already mentioned, I have tried not to be too formal, and the price for that is that things may sometimes lack precision. I have tried to give all needed references, and for the basics, I will mainly refer to the following books. The book of A. Henrot [He] constitutes a very good reference for the study of the spectrum of the Laplacian for Euclidean domains, and the books of I. Chavel [Ch1] and P. Bérard [Be] are very convenient for the spectrum of the Laplacian on Riemannian manifolds. These two last books, which were published when I was a graduate student, have become dated. Nevertheless, they contain a lot of basic and important results. More recent references are the book of Rosenberg [Ro] (but rather far from the goal of this lecture) and the lecture notes about Spectral Theory and Geometry edited by B. Davies and Y. Safarov [ST]. Last but not least, it is always interesting to look at the famous book of Berger-Gauduchon-Mazet [BGM], who played an important role in the development of the Geometrical Spectral Theory. At the moment when the summer school took place, the book by O. Lablée [La] appeared. I did not use it to write these notes but it is an interesting introduction to the subject.

**Acknowledgments.** I thank the organizers of the summer school (especially Alexandre Girouard and Iosif Polterovich) for inviting me to Montréal and for the opportunity to write this article.

## 1. Introduction, basic results and examples

In this section, I will describe without proof some classical results and examples concerning the spectrum of the Laplacian on domains and on Riemannian manifolds. Most of

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the time, I will introduce the different notions rather informally and refer to classical texts for more details. The aim is to give a very rough first idea of a part of the "landscape" of the geometrical spectral theory thanks to a couple of examples and results. It is as well an opportunity to introduce some of the other lectures of the summer school.

### 1.1. The Laplacian for Euclidean domains with Dirichlet boundary conditions.

Let $\Omega \subset \mathbb{R}^n$ be a bounded, connected domain with Lipschitz boundary. The *Laplacian* we will consider is given by

$$\Delta f = -\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$$

where $f \in C^2(\Omega)$. We investigate the spectrum of the Laplacian $\Delta$ on $\Omega$ with the *Dirichlet boundary condition*, that is we study the eigenvalue problem

$$\Delta f = \lambda f \tag{1}$$

under the Dirichlet condition

$$f_{|\partial\Omega} = 0. \tag{2}$$

This problem has a discrete and real spectrum

$$0 < \lambda_1(\Omega) < \lambda_2(\Omega) \leq \lambda_3(\Omega) \leq ... \to \infty,$$

with the eigenvalues repeated according to their multiplicity.

**Example 1.** *If $\Omega$ is the interval $]0,\pi[\subset \mathbb{R}$, the Laplacian is given by $\Delta f = -f''$ and the spectrum, for the Dirichlet boundary condition, is given by $\lambda_k = k^2$, $k = 1, 2, ...$ The eigenfunction corresponding to $\lambda_k = k^2$ is $f_k(x) = \sin kx$.*

However, it is exceptional to be able to calculate explicitly the spectrum. In dimensions greater than one, it is generally not possible (apart from a few exceptions like the ball or a product of intervals) to calculate the spectrum of a domain. The calculation of the spectrum of a ball is classic, but not easy. It is done with enough details, for example, in the introduction of the Ph. D. thesis of A. Berger [Ber].

Given that, physically, the Dirichlet problem is a modelization of the *vibration of a fixed membrane*, it can be hoped to have connections between the *geometry* (or *shape*) of a domain $\Omega$ and its spectrum. By *fixed* membrane is understood a membrane whose boundary is fixed, like a drum. This leads to the following very classical image for two types of problems in the context of geometrical spectral theory:

**The direct problems.** When we see a drum, we have an idea of how it would sound. The mathematical translation is: if we have some information about the geometry (or the shape) of a domain $\Omega$, we can hope to get information (or estimates) about its spectrum. By *information about the geometry* can be understood for example the volume, the diameter of $\Omega$, the mean curvature of its boundary, the inner radius. By *estimates of the spectrum*, we understand lower or upper bounds for some or all of the eigenvalues, with respect to the geometrical information we have.

**The inverse problems.** If we hear a drum, even if we do not see it, we have an idea of its shape. The mathematical translation is: If we have information about the spectrum of a domain, we can hope to get information about its geometry. The most famous question is whether the shape of a domain is determined by its spectrum (question of M. Kac in 1966: can one hear the shape of a drum?). In other words: if we know all the spectrum of a domain, are we able to reconstitute the domain? The formal answer is no: There are a lot of domains having the same spectrum without being isometric. However, in some sense, this is often true, which makes things complicated. I will not go further in this direction in these notes, see [Go] for a survey of the subject.

1.1.1. *The Faber-Krahn inequality.* This is an old but very enlightening result found independentely by Faber and Krahn in the 1920's: More explanations and a sketch of the proof can be found in [As] p.102-103. The proof uses the isoperimetric inequality and is not easy. In his lecture, D. Bucur will offer an approach for a new proof of this inequality.

**Theorem 2.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $B \subset \mathbb{R}^n$, a ball with the same volume as $\Omega$. Then, we have*

$$\lambda_1(B) \leq \lambda_1(\Omega),$$

*with equality if and only if $\Omega$ is equal to $B$ up to a displacement.*

This theorem first gives a lower bound for the first Dirichlet eigenvalue of a domain $\Omega$ which is a typical direct problem. If the volume of $\Omega$ is known, then one gets immediately information about the spectrum: a lower bound for the first eigenvalue. But the *equality case* can be understood as an inverse result. If $\Omega$ and $B$ have the same volume and if $\lambda_1(\Omega) = \lambda_1(B)$, then $\Omega$ is a ball.

This result is also very interesting because it allows very modern questions to be asked. We will mention a few of them, without going into much detail. The same type of questions could also be asked in other situations (other boundary conditions, Laplacian on Riemannian manifolds, other differential operators, etc.).

(1) **The stability.** If a domain $\Omega \subset \mathbb{R}^n$ has the same volume as a ball $B$ and if, moreover, $\lambda_1(\Omega)$ is close to $\lambda_1(B)$, can one say that $\Omega$ is *close* to the ball $B$? Of course, a difficulty is to state what is meant by "close". There are a lot of results around this problem that I will not describe in detail, see [FMP] for a recent and important contribution.

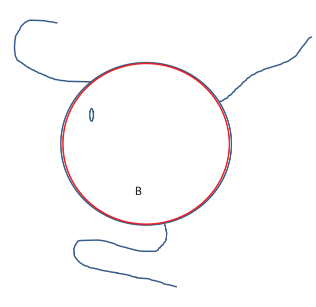



FIGURE 1. The domain $\Omega$ is close to the ball $B$ in the sense of the measure

Note that it is not too difficult to see that one can do a small hole in any domain or add a “thin hair” without affecting the spectrum too much (easy part in Example 16 and more explanations in [BC], [CGI]). Therefore, it cannot be hoped for example, that $\Omega$ will be homeomorphic to a ball, or will be Hausdorff close to a ball. However, essentially, one can show that $\Omega$ is close to a ball in the $L^2$-*sense*, see Figure 1.

(2) **The spectral stability.** The next natural question is to see what this stability will imply: in [BC], we have shown that if $\Omega$ and $B$ have the same volume, and if $\lambda_1(\Omega)$ is close to $\lambda_1(B)$ then, for all $k$, $\lambda_k(\Omega)$ is close to $\lambda_k(B)$ (but, of course, not uniformly in $k$). The difficulty is that the proximity of $\lambda_1(\Omega)$ and $\lambda_1(B)$ implies only that the domains are close in a $L^2$ sense, and we have to show that this is enough to control all the eigenvalues.

(3) **The extremal domains.** We have seen that, among all domains having the same volume, the ball is the domain having the minimal $\lambda_1$. We can of course ask for domain(s) having the minimal $\lambda_k$. The various aspects of this problem are very complicated and most are far from being solved. For $\lambda_2$, it is well known that the solution is the union of two disjoint balls (see [He] thm. 4.1.1). Note that this domain is not regular in the sense that was expected. But in general, it is already difficult to find a right context where it can be shown that an extremal domain exists. It will be one of the goals of D. Bucur's lecture to explain this, and I do not develop the question more here.

It is interesting to sketch the proof for $\lambda_2$ as it is done, for example, in [He]. It uses the Courant nodal Theorem (see the lecture of B. Helffer or [Ch1] p. 19). The first implication of this theorem is that the eigenfunction $f_1$ associated to the first eigenvalue $\lambda_1$ of a domain $\Omega$ has a constant sign (say it is positive) and has multiplicity one. The second eigenvalue $\lambda_2$ of $\Omega$ could have multiplicity. If $f_2$ is an eigenfuction associated to $\lambda_2$, it has to be orthogonal to $f_1$, and, as consequence, change its sign. Let

$$A_1 = \{x \in \Omega : f_2(x) > 0\};\ A_2 = \{x \in \Omega : f_2(x) < 0\}.$$

The Courant nodal theorem says that $A_1$ and $A_2$ are connected. Thanks to this information and to the theorem of Faber-Krahn, we can prove that the extremal domain for $\lambda_2$ is the union of two ball of the same size. Let $\Omega$ be a domain, and $f_2$ an eigenfunction for $\lambda_2$.

The restriction of $f_2$ to $A_1$ and to $A_2$ satisfies the Laplace equation, and $f_2$ is an eigenfunction of eigenvalue $\lambda_2$. But on $A_1$ and $A_2$, $f$ does not change its sign. This mean that $f_2$ is an eigenfunction for the *first* eigenvalue of $A_1$ and $A_2$, and it follows

$$\lambda_1(A_1) = \lambda_1(A_2) = \lambda_2(\Omega).$$

Let $B_1, B_2$ two balls with $Vol(B_1) = Vol(A_1)$ and $Vol(B_2) = Vol(A_2)$. By Faber-Krahn, we have

$$\lambda_1(A_i) \geq \lambda_1(B_i),\ i = 1, 2,$$

with equality if $A_i$ is congruent to $B_i$, so that $\lambda_2(\Omega) \geq max(\lambda_1(B_1), \lambda_1(B_2))$.

If we consider the disjoint union $B^*$ of $B_1$ and $B_2$, the spectrum of $B^*$ is the union of both spectrum, and $Vol(B^*) = Vol(\Omega)$, so that $\lambda_2(B^*) \leq max(\lambda_1(B_1), \lambda_1(B_2))$.

We deduce $\lambda_2(B^*) = max(\lambda_1(B_1), \lambda_1(B_2))$. In order to have the smallest possible $\lambda_2$, we have to choose two balls of the same radius, and the equality case in the Faber-Krahn inequality implies that $\Omega$ is the disjoint union of two identical balls.

(4) **Numerical estimates.** Knowing about the existence of extremal domains from a theoretical viewpoint, one can try to investigate *the shape* of such domains. It is almost impossible to do this from a formal point of view, and this was investigated from a numerical point of view, in particular for domains of $\mathbb{R}^2$. But this is also a difficult numerical problem, in particular due to the fact that there are a lot of *local* extrema. This approach was initiated in the thesis of E. Oudet. Note that it gives us a candidate to be extremal, but we have no proof that the domain appearing as extremal is the right one. It is important to apply different numerical approaches, and to see if they give the same results. See the thesis of A. Berger [Ber] for a description of results.

**Remark 3.** *A question could be whether a domain of given fixed volume could have a very large first eigenvalue $\lambda_1$: The answer is yes. For example, think of a long and thin domain, and more generally of "thin" domains, that is domains with small inradius. However, formally, a small inradius does not imply large eigenvalue $\lambda_1$. We have to take into account the* connectivity *of the domain, see [Cr]. For example, a ball with a lot of very small holes can have a first eigenvalue close to the first eigenvalue of the ball (the problem of the Crushed Ice described for example in [Ch1] p. 233).*

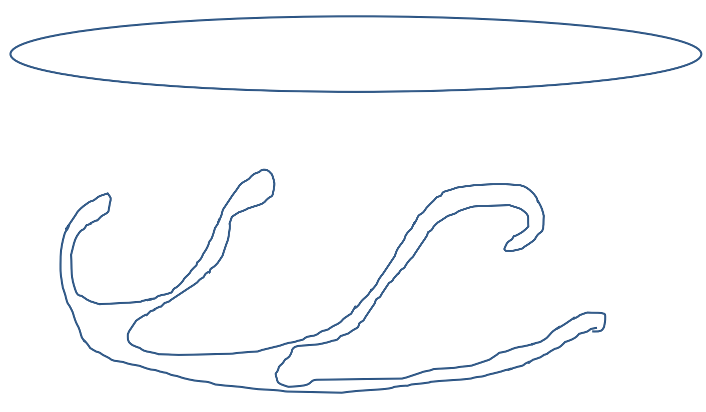


FIGURE 2. Examples with "large" Dirichlet eigenvalue

### 1.2. **The Laplacian for Euclidean domains with Neumann boundary conditions.**

Let $\Omega \subset \mathbb{R}^n$ be a bounded, connected domain with a Lipschitz boundary. The *Laplacian* is given by

$$\Delta f = -\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$$

where $f \in C^2(\Omega)$. What is investigated is the spectrum of the Laplacian $\Delta$ on $\Omega$ with the *Neumann boundary condition*, that is the eigenvalue problem

$$\Delta f = \lambda f \tag{3}$$

under the Neumann condition

$$\left(\frac{\partial f}{\partial n}\right)_{|\partial\Omega} = 0. \tag{4}$$

where $n$ denotes the exterior normal to the boundary.

This problem has a discrete and real spectrum

$$\lambda_0 = 0 < \lambda_1(\Omega) < \lambda_2(\Omega) \leq \lambda_3(\Omega) \leq \ldots \to \infty,$$

with the eigenvalues repeated according to their multiplicity.

**Example 4.** *If $\Omega$ is the interval $]0, \pi[\subset \mathbb{R}$, the Laplacian is given by $\Delta f = -f''$ and the spectrum, for the Neumann boundary condition, is given by $\lambda_k = k^2$, $k = 0, 1, 2, \ldots$ The eigenfunction corresponding to $\lambda_k = k^2$ is $f_k(x) = \cos kx$.*

We also refer to [Ber] for the calculation of the spectrum of the ball in dimensions 2 and 3.

Some similarities between the spectrum for the Neumann boundary condition and Dirichlet boundary condition can be noticed. However, in this introduction, I will mainly focus on the differences.

A first obvious difference is that zero is always an eigenvalue, corresponding to the constant function. This means that the first *interesting* eigenvalue is the second, or the first *nonzero* eigenvalue.

In Example 13, for the Dirichlet problem, we will see that if $\Omega_1 \subset \Omega_2$, then we have for each $k$ that $\lambda_k(\Omega_1) \geq \lambda_k(\Omega_2)$ (we say that we have *monotonicity*). On the contrary, in Example 18, we will see that we have no monotonicity for the Neumann problem. Indeed, in this case, the fact that $\Omega_1 \subset \Omega_2$ has no implication on the spectrum.

We will also see that the spectrum for the Neumann conditions is *very* sensitive to perturbations of the boundary. At this stage, I must clarify that, because the goal of these lectures was to present a geometric approach, I have chosen bounded domains with a *Lipschitz* boundary. This guarantees the discreteness of the spectrum, but this hypothesis may be relaxed, in particular for the Dirichlet problem (see the lectures of D. Bucur). In Example 19, we will construct a bounded domain, with a smooth boundary up to one singular point, and a non-discrete spectrum.

1.2.1. *The Szegö-Weinberger inequality.* For further details, please refer to [As] p. 103. It is similar to the Faber-Krahn inequality, but is more delicate to prove, precisely because it concerns the second eigenvalue $\lambda_1$.

**Theorem 5.** *Let $\Omega \subset \mathbb{R}^n$ a domain and $B \subset \mathbb{R}^n$, a ball with the same volume of $\Omega$. Then, we have*

$$\lambda_1(\Omega) \leq \lambda_1(B),$$

*with equality if and only if $\Omega$ is equal to $B$ up to a displacement.*

**Remark 6.**
(1) *In contrast with the case of the Dirichlet boundary condition, $\lambda_1$ cannot be very large for a domain of fixed volume when we take the Neumann boundary condition.*
(2) *We will see that there are domains of fixed volume with as many eigenvalues close to $0$ as we want.*
(3) *There is no spectral stability in the Neumann problem. Indeed, we can perturb a ball by pasting a thin full cylinder of convenient length $L$: It does not affect the first nonzero eigenvalue but all the others. Using results in the spirit of [An1], the spectrum converges to the union of the spectrum of the ball and of the spectrum of $[0, L]$, with the Dirichlet boundary condition on $0$ and the Neumann condition on $L$. It suffices to choose $L$ in such a way that the first eigenvalue of the interval lies between the two first eigenvalues (without multiplicity) of the ball.*

1.3. **The Laplacian on a Riemannian manifold.** Let $(M, g)$ be a Riemannian manifold. If $f \in C^2(M)$, then let us define

$$\Delta f = -div\ grad\ f,$$

and look at the eigenvalue problem

$$\Delta f = \lambda\ f.$$

In local coordinates $\{x_i\}$, the Laplacian reads

$$\Delta f = -\frac{1}{\sqrt{det(g)}} \sum_{i,j} \frac{\partial}{\partial x_j}(g^{ij}\sqrt{det(g)}\frac{\partial}{\partial x_i} f).$$

In particular, in the Euclidean case, we recover the usual expression

$$\Delta f = -\sum_j \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j} f.$$

We will not use very often the expression of the Laplacian in local coordinates. People who are not familiar with the Riemannian geometry could think of a Riemannian manifold as a submanifold of the Euclidean space. Most of the operations we will make consist in "deforming" the Riemannian metric $g$ in order to construct examples or counter-examples. These deformations can be viewed as deformations of the submanifold. However, this intuitive way of thinking has its limits: Some deformations are much easier to realize abstractly as deformations of the Riemannian metric, than to be viewed directly as deformations of a submanifold. This is the case of the *conformal deformation* which will be considered in the second part of this lecture.

When $M$ is compact and connected, we have a discrete spectrum

$$0 = \lambda_0(M, g) < \lambda_1(M, g) \leq \lambda_2(M, g) \leq ... \to \infty.$$

The eigenvalue 0 corresponds to the constant eigenfunction.

**Example 7.** *Let us take for $M$ the circle of radius 1 that we can identify with the interval $[0, 2\pi]$ with periodic condition. The spectrum is given by $\lambda_0 = 0$ and*

$$\lambda_{2k-1} = \lambda_{2k} = k^2$$

*for $k \geq 1$ corresponding to the eigenfuctions $f_{2k-1}(x) = \sin(kx)$, $f_{2k}(x) = \cos(kx)$.*

Here as well, there are very few situations where the spectrum can be explicitly calculated. However, there are more examples in the context of Riemannian manifolds than for domains. In the book [BGM], the case of the n-dimensional round sphere, and of the flat tori in dimension 2 are studied.

Note that, in dimension greater than 1, the eigenvalues are generically of multiplicity 1 [Uh]. However, in the very few examples where the spectrum can be calculated, large multiplicity often arises. This comes from the fact that the examples where we can calculate explicitly the spectrum have a lot of symmetries, and this implies very often the emergence of multiplicity.

1.3.1. *Large and small eigenvalues.* We can address a similar question as that asked about domains. Given a compact connected differential manifold $M$ without boundary, is it possible to find a metric $g$ on $M$ with $\lambda_1(M, g)$ arbitrarily large or small? First, we have to observe that some constraint is needed: for any Riemannian metric $g$, we have the relation

$$\lambda_k(M, t^2 g) = \frac{1}{t^2}\lambda_k(M, g).$$

This means that, thanks to a homothety of the metric, arbitrarily large or small eigenvalues can be produced. This is also true for Euclidean domains.

A possible constraint is to fix the volume, or, equivalently, to estimate $\lambda_k(M, g)Vol(M, g)^{2/n}$ rather than $\lambda_k(M, g)$ , where $n = dim\ M$, because this expression is invariant by homothety.

**Small eigenvalues.** We will soon see that it is easy to produce a metric of given volume with as many small eigenvalues as we want. This leads to the question of finding lower bounds depending on the geometry.

**Large eigenvalues.** For large eigenvalues, the result is surprising and it leads to a lot of different types of questions.

**The case $dimM \geq 3$.** If $M$ is a given compact differential manifold with $dim\ M \geq 3$, then it was proved in [CD] that we can choose $g$ of given volume with $\lambda_1$ as large as we want. To prove this result:

- first, prove this for the sphere of odd dimension (see for example [Bl]) using that the sphere of dimension $(2n + 1)$ is a $S^1$-bundle over $\mathbb{C}P^n$.
- extend the result to the even dimension sphere [Mu].
- prove it for every manifold $M$ of dimension $\geq 3$ [CD], using classical, but rather difficult results from spectral theory. The idea is to deform the metric of the manifold to make it "close" to the metric of a sphere with large eigenvalue, and then to conclude using results about the behavior of the spectrum under surgery.

**The case of surfaces**. For surfaces, there is an upper bound for $\lambda_k$ which depends on the genus and on $k$. This will be proven in the second part of this lecture. But the case of the first nonzero eigenvalue $\lambda_1$ is of special interest, and I will conclude this first section with some consideration about this question.

1.3.2. *The case of the first nonzero eigenvalue for surfaces.* For this paragraph, please refer to the recent survey of R. Schoen [Sc] paragraph 3, where one can find precise references and explanations, and to [CE1].

A first result was proved by Hersch for the sphere in 1970 [He]: for any Riemannian metric $g$ on the sphere $S^2$

$$\lambda_1(S^2, g)Vol(S^2, g) \leq 8\pi \tag{5}$$

with equality if and only if $g$ is the canonical round metric. In 1980, Yang and Yau extended the result to every genus: if $S$ is an orientable surface of genus $\gamma$ and $g$ a Riemannian metric on $S$

$$\lambda_1(S, g)Vol(S, g) \leq 8\pi E(\frac{\gamma + 3}{2}) \tag{6}$$

where $E$ denotes the integer part. This result is good, in the following sense: There exists a universal constant $C > 0$ such that, for each $\gamma$, there exists a surface $(S, g)$ of genus $\gamma$ with $\lambda_1(S, g)Vol(S, g) \geq C\gamma$ (and this is not easy at all to construct). However, it is not optimal, in the sense that there is no equality case.

There are only a few surfaces where the maximum/extremum of $\lambda_1(S, g)Vol(S, g)$ is known:

- the projective space, where the maximum is $12\pi$, reached for the metric with constant curvature;
- the torus, where the maximum is $\frac{8\pi^2}{\sqrt{3}}$, reached by the flat equilateral torus;
- the Klein bottle, where the maximum is $12\pi E(\frac{2\sqrt{2}}{3})$ with an equality case.
- the surface of genus 2. In this situation, the maximum is $16\pi$ and is reached by a family of singular metrics. At some point in the proof, there is a numerical estimate.

For surfaces with genus greater than 2, almost nothing is known about maximum/supremum for $\lambda_1$.

## 2. Variational characterization of the spectrum and simple applications

In this section, I give a variational characterization of the spectrum: Roughly speaking, this allows us to investigate the spectrum of the Laplacian without looking at the equation $\Delta u = \lambda u$ itself, but rather by considering some *test functions*. Moreover, we have only to take into account the gradient of these test functions, and not on their second derivative, which is much easier. One can say that this variational characterization is one of the fundamental tools allowing us to use the geometry in order to investigate the spectrum. To illustrate this, I will explain with some detail very elementary constructions *of small eigenvalues*: These constructions are well known by the specialists of the topic, and, in general, they are mentioned in one sentence. Although these are used here at an elementary level, it allows us to understand principles that will be used later in a more difficult context. Please, refer to the book of P. Bérard [Be] for more details about the theory.

Let $(M,g)$ be a compact manifold with boundary $\partial M$. Let $f,h \in C^2(M)$. Then we have the well-known Green's Formula

$$\int_M \Delta f h dvol_g = \int_M \langle df, dh\rangle dvol_g - \int_{\partial M} h\frac{df}{dn}dA$$

where $\frac{df}{dn}$ denotes the derivative of $f$ in the direction of the outward unit normal vector field $n$ on $\partial M$. $dvol_g$ the volume form on $(M,g)$ (associated to $g$) and $dA$ the volume form on $\partial M$.

In particular, if one of the following conditions $\partial M = \emptyset$, $h_{|\partial M} = 0$ or $(\frac{df}{dn})_{|\partial M} = 0$ is satisfied, then we have the relation

$$(\Delta f, h) = (df, dh) = (f, \Delta h)$$

where $(,)$ denotes the $L^2$-scalar product.

In the sequel, we will study the following eigenvalue problems when $M$ is compact:

- Closed Problem:
$$\Delta f = \lambda f \ in \ M; \ \partial M = \emptyset;$$
- Dirichlet Problem
$$\Delta f = \lambda f \ in \ M; \ f_{|\partial M = 0};$$
- Neumann Problem:
$$\Delta f = \lambda f \ in \ M; \ (\frac{df}{dn})_{|\partial M} = 0.$$

The following standard result about the spectrum can be seen in [Be] p. 53.

**Theorem 8.** *Let $M$ be a compact connected manifold with boundary $\partial M$ (which may be empty), and consider one of the above mentioned eigenvalue problems. Then:*

1. *The set of eigenvalue consists of an infinite sequence $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq ... \to \infty$, where $0$ is not an eigenvalue in the Dirichlet problem;*
2. *Each eigenvalue has finite multiplicity and the eigenspaces corresponding to distinct eigenvalues are $L^2(M)$-orthogonal;*
3. *The direct sum of the eigenspaces $E(\lambda_i)$ is dense in $L^2(M)$ for the $L^2$-norm. Furthermore, each eigenfunction is $C^\infty$-smooth.*

**Remark 9.** *The Laplace operator depends only on the given Riemannian metric. If*

$$F : (M,g) \to (N,h)$$

*is an isometry, then $(M,g)$ and $(N,h)$ have the same spectrum, and if $f$ is an eigenfunction on $(N,h)$, then $f \circ F$ is an eigenfunction on $(M,g)$ for the same eigenvalue.*

As already mentioned, the spectrum cannot be computed explicitly, with a few exceptions. In general, it is only possible to get estimates of the spectrum, and these estimations are related to the geometry of the considered manifold $(M,g)$. However, asymptotically, we know how the spectrum behaves. This is the Weyl law.

**Weyl law:** If $(M,g)$ is a compact Riemannian manifold of dimension $n$ (with or without boundary), then, for each of the above eigenvalue problems,

$$\lambda_k(M,g) \sim \frac{(2\pi)^2}{\omega_n^{2/n}} \left( \frac{k}{Vol(M,g)} \right)^{2/n} \tag{7}$$

as $k \to \infty$, where $\omega_n$ denotes the volume of the unit ball of $\mathbb{R}^n$.

This also means

$$\lim_{k\to\infty} \frac{\lambda_k(M,g)}{k^{2/n}} = \frac{(2\pi)^2}{\omega_n^{2/n}} \left( \frac{1}{Vol(M,g)} \right)^{2/n} \tag{8}$$

In particular, the spectrum of $(M,g)$ determines the volume of $(M,g)$.

It is important to stress that the result is asymptotic: We do not know in general for which k the asymptotic estimate becomes good! However, this formula is a guide when trying to get upper bounds.

To introduce some geometry on the study of the Laplacian, it is very relevant to use the variational characterization of the spectrum. To this aim, let us introduce the Rayleigh quotient. In order to do this, we have to consider functions in the *Sobolev space* $H^1(M)$ or $H_0^1(M)$. It is possible to find the definition of $H^1(M)$ or $H_0^1(M)$ in every book of PDE, but what we will really consider are functions which are of class $C^1$ by part and glue them continuously along a submanifold. Moreover, for $H_0^1(M)$, we consider only functions taking the value 0 on $\partial M$.

If a function $f$ lies in $H^1(M)$ in the closed and Neumann problems, and in $H_0^1(M)$ for the Dirichlet problem, the Rayleigh quotient of $f$ is

$$R(f) = \frac{\int_M |df|^2 dvol_g}{\int_M f^2 dvol_g} = \frac{(df,df)}{(f,f)}.$$

Note that in the case where $f$ is an eigenfunction for the eigenvalue $\lambda_k$, then

$$R(f) = \frac{\int_M |df|^2 dvol_g}{\int_M f^2 dvol_g} = \frac{\int_M \Delta f \ f dvol_g}{\int_M f^2 dvol_g} = \lambda_k.$$

**Theorem 10.** *(Variational characterization of the spectrum, [Be] p. 60-61.) Let us consider one of the 3 eigenvalues problems.*

***Min-Max formula:*** *we have*

$$\lambda_k = \inf_{V_k} sup\{R(u) : u \neq 0,\ u \in V_k\}$$

*where $V_k$ runs through $k+1$-dimensional subspaces of $H^1(M)$ ($k$-dimensional subspaces of $H_0^1(M)$ for the Dirichlet eigenvalue problem).*

This min-max formula is useless to calculate $\lambda_k$, but it is very useful to find an upper bound because of the following fact: For any given $(k+1)$ dimensional vector subspace $V$ of $H^1(M)$, we have

$$\lambda_k(M,g) \leq sup\{R(u) : u \neq 0,\ u \in V\}.$$

This gives immediately an upper bound for $\lambda_k(M,g)$ if it is possible to estimate the Rayleigh quotient $R(u)$ of all the functions $u \in V_k$. Note that there is no need to calculate the Rayleigh quotient, it suffices to estimate it from above. Of course, this upper bound is useful if the vector space $V$ is conveniently chosen.

It would be a delicate problem to estimate the Rayleigh quotient of all functions $u \in V$. A special situation is when $V$ is generated by $k+1$ *disjointly supported functions* $f_1, ..., f_{k+1}$. In this case,

$$sup\{R(u) : u \neq 0,\ u \in V\} = sup\{R(f_i) : i = 1, ..., k+1\}, \tag{9}$$

which makes the estimation easier to do, because we have only to estimate from above the Rayleigh quotient of $(k+1)$ functions. We will use this fact in the sequel.

To see it, let $f = \alpha_1 f_1 + ... + \alpha_{k+1} f_{k+1}$. Then

$$(f, f) = \alpha_1^2 (f_1, f_1) + ... + \alpha_{k+1}^2 (f_{k+1}, f_{k+1}).$$

and

$$(df, df) = \alpha_1^2 (df_1, df_1) + ... + \alpha_{k+1}^2 (df_{k+1}, df_{k+1}).$$

so that $R(f)$ is bounded from above by the maximum of the Rayleigh quotients $R(f_i)$, $i = 1, ..., k+1$.

**Remark 11.** *We have indeed shown a little more: If the test functions $\{f_i\}_{i=1}^{k+1}$ are orthonormal* and *if the $\{df_i\}_{i=1}^{k+1}$ are orthogonal, then (9) is also true.*

**Remark 12.** *We can see already two advantages to this variational characterization of the spectrum. First, there is no need to work with solutions of the Laplace equation, but only with "test functions", which is easier. Then, only one derivative of the test function has to be controlled, and not two, as in the case of the Laplace equation.*

To see this concretely, let us give with some details a couple of simple examples.

**Example 13.** ***Monotonicity in the Dirichlet problem.*** *Let $\Omega_1 \subset \Omega_2 \subset (M, g)$, two domains of the same dimension $n$ of a Riemannian manifold $(M, g)$. Let us suppose that $\Omega_1$ and $\Omega_2$ are both compact connected manifolds with a smooth boundary. If we consider the Dirichlet eigenvalue problem for $\Omega_1$ and $\Omega_2$ with the induced metric, then for each $k$*

$$\lambda_k(\Omega_2) \leq \lambda_k(\Omega_1).$$

The proof is very simple: Each eigenfunction of $\Omega_1$ may be continuously extended by 0 on $\Omega_2$ and may be used as a test function for the Dirichlet problem on $\Omega_2$.

Let us construct an upper bound for $\lambda_k(\Omega_2)$: for $V_k$, we choose the vector subspace of $H_0^1(\Omega_2)$ generated by an orthonormal basis $f_1, ..., f_k$ of eigenfunctions of $\Omega_1$ extended by 0 on $\Omega_2$. Clearly, these functions vanish on $\partial\Omega_2$ and they are $C^\infty$ on $\Omega_1$ and $\Omega_2 \cap \bar{\Omega_1}^c$. They are continuous on $\partial\Omega_1$.

Let $f = \alpha_1 f_1 + ... + \alpha_k f_k$. We have

$$(f, f) = \alpha_1^2 + ... + \alpha_k^2.$$

Because $(df_i, df_i) = (\Delta f_i, f_i) = \lambda_i(\Omega_1)(f_i, f_i)$ and $(df_i, df_j) = (\Delta f_i, f_j) = 0$ if $i \neq j$, we have

$$(df, df) = \alpha_1^2 \lambda_1(\Omega_1) + ... + \alpha_k^2 \lambda_k(\Omega_1).$$

We conclude that $R(f) \leq \lambda_k(\Omega_1)$, and we have $\lambda_k(\Omega_2) \leq \lambda_k(\Omega_1)$. As this is true for each $k$, we have the result.

**Example 14.** *As a consequence of (9), if $M$ is a compact Riemannian manifold without boundary, and if $\Omega_1,...,\Omega_{k+1}$ are domains in $M$ with disjoint interiors, then*

$$\lambda_k(M,g) \leq max(\mu_1(\Omega_1),...,\mu_1(\Omega_{k+1})),$$

*where $\mu_1(\Omega)$ denotes the first eigenvalue of $\Omega$ for the Dirichlet problem.*

*Let us choose $(k+1)$ disjointly supported test functions:*

*We consider the eigenfunctions $u_1,...u_{k+1}$ corresponding to $\mu_1(\Omega_1),...,\mu_1(\Omega_{k+1})$ respectively, extended by $0$ on $M$.*

*They have $\mu_1(\Omega_1),...,\mu_1(\Omega_{k+1})$ as Rayleigh quotients, and this allows us to show the inequality.*

**Example 15.** *Let us see how to deform a Riemannian manifold (keeping the volume fixed), in order to produce as many small eigenvalues as we want.*

*We first consider the cylinder $C_L = [0,L] \times S^{n-1}$, where the first Dirichlet eigenvalue satisfies*

$$\lambda_1(C_L) \leq \frac{\pi^2}{L^2}$$

*It suffices to consider the function*

$$f(r,x) = f(r) = \sin\frac{\pi r}{L}$$

*which is an eigenfunction for $\frac{\pi^2}{L^2}$.*

*Then, on each manifold $M$, we can construct a family of Riemannian metrics of fixed volume, with as many small eigenvalues as we want. We just deform the metric without changing the topology as follows: we deform locally a small ball into a long and thin "nose", that is a cylinder $[0,L] \times S^{n-1}_\epsilon$ closed by a cap. We can divide this cylindrical part into $(k+1)$ disjoint cylinders of length $\frac{L}{k+1}$, and, considering the Dirichlet problem on each of these cylinders, we get*

$$\lambda_k \leq \frac{\pi^2(k+1)^2}{L^2}$$

*which is arbitrarily small for $k$ fixed and $L \to \infty$.*

In the sequel, we will explain how to produce arbitrarily small eigenvalues for a Riemannian manifold with fixed volume and *bounded diameter* (Cheeger's dumbbell construction and its applications). But let us first apply the min-max formula to an elementary situation of *surgery*.

**Example 16.** ***Elementary surgery.*** *We will explain in detail the already mentioned fact:* A small hole in a domain does not affect the spectrum "too much"(see (10) below). *Let $\Omega \subset \mathbb{R}^n$ a domain with smooth boundary. Let $x \in \Omega$, and $B(x,\epsilon)$ the ball of radius $\epsilon$ centered at the point $x$ ($\epsilon$ is chosen small enough such that $B(x,\epsilon) \subset \Omega$). We denote by $\Omega_\epsilon$ the subset $\Omega - B(x,\epsilon)$, that is we make a hole of radius $\epsilon$ on $\Omega$.*

*We consider the two domains $\Omega$ and $\Omega_\epsilon$ with the Dirichlet boundary conditions, and denote by $\{\lambda_k\}_{k=1}^\infty$ and by $\{\lambda_k(\epsilon)\}_{k=1}^\infty$ their respective spectrums.*

*Then, for each $k$, we have*

$$\lim_{\epsilon \to 0} \lambda_k(\epsilon) = \lambda_k \tag{10}$$

Before showing this in detail, let us make a few remarks.

**Remark 17.** (1) *The convergence is not uniform in $k$.*
(2) *This result is a very specific part of a much more general facts. We get this type of results on manifolds, with subset much more general than balls, for example tubular neighborhood of submanifolds of codimension greater than 1. For the interested reader, we refer to the paper of Courtois [Cou] and to the book of Chavel [Ch1].*
(3) *We can get very precise asymptotic estimates of $\lambda_k(\epsilon)$ in terms of $\epsilon$. Again, we refer to [Cou] and references therein.*
(4) *We have also the convergence of the eigenspace associated to $\lambda_k(\epsilon)$ to the eigenspace associated to $\lambda_k$, but this has to be defined precisely: a problem occurs if the multiplicity of $\lambda_k$ is not equal to the multiplicity of $\lambda_k(\epsilon)$, see [Cou].*

In order to see the convergence, we first observe that by monotonicity, for all $k$

$$\lambda_k \leq \lambda_k(\epsilon).$$

Our goal is to show that

$$\lambda_k(\epsilon) \leq \lambda_k + c(\epsilon),$$

with $c(\epsilon) \to 0$ as $\epsilon \to 0$.

In order to use the min-max construction, we need to consider a family of test-functions. This family is constructed using a perturbation of the eigenfunctions of $\Omega$. Note that we do the proof for $n \geq 3$, but the strategy is the same if $n = 2$.

Let $\{f_i\}_{i=1}^{\infty}$ be an orthonormal basis of eigenfuctions on $\Omega$. We fix a $C^\infty$-plateau function $\chi : \mathbb{R} \to \mathbb{R}$ defined by

$$\chi(r) = \begin{cases} 0 & \text{if } r \leq \frac{3}{2} \\ 1 & \text{if } r \geq 2 \\ 0 \leq \chi(r) \leq 1 & \text{if } \frac{3}{2} \leq r \leq 2 \end{cases}$$

The test function $f_{i,\epsilon}$ associated to $f_i$ is defined by

$$f_{i,\epsilon}(p) = \chi(\frac{d(p,x)}{\epsilon}) f_i.$$

By construction, the function $f_{i,\epsilon}(p)$ takes the value 0 on the ball $B(x,r)$ and satisfies the Dirichlet boundary condition. So, it can be used as a test function in order to estimate $\lambda_{k,\epsilon}$.

Let $f$ be a function on the vector space generated by $\{f_{i,\epsilon}\}_{i=1}^{k}$. In order to estimate its Rayleigh quotient, it makes a great simplification if the basis is orthonormal and the $(df_i)$ are orthogonal. This is not the case here. However, the basis is *almost* orthonormal and the $(df_i)$ are almost orthogonal: this is what we will first show and use. Note that this is a very common situation.

**The basis $\{f_{i,\epsilon}\}_{i=1}^k$ is almost orthonormal.** Let us denote by $a_i$ (resp. $b_i$) the maximum of the function $|f_i|$ (resp. $|df_i|$) on $\Omega$. Then

$$\int_{B(x,2\epsilon)} f_i^2 \leq a_i^2 2^n \omega_n \epsilon^n, \ \int_{B(x,2\epsilon)} |df_i^2|^2 \leq b_i^2 2^n \omega_n \epsilon^n \tag{11}$$

where $\omega_n$ is the volume of the unit ball in $\mathbb{R}^n$.

Let us show that

$$|\langle f_{i,\epsilon}, f_{j,\epsilon}\rangle| \leq \delta_{ij} + C_k \epsilon^{n/2},$$

where $C_k$ depends on $a_1, ..., a_k$.

$$\langle f_{i,\epsilon}, f_{j,\epsilon}\rangle = \langle \chi_\epsilon f_i, \chi_\epsilon f_j\rangle = \langle (\chi_\epsilon - 1)f_i + f_i, (\chi_\epsilon - 1)f_j + f_j\rangle =$$
$$= \langle f_i, f_j\rangle + \langle (\chi_\epsilon - 1)f_i, f_j\rangle + \langle (\chi_\epsilon - 1)f_j, f_i\rangle + \langle (\chi_\epsilon - 1)f_i, (\chi_\epsilon - 1)f_j\rangle.$$

By (11), $\|(\chi_\epsilon - 1)f_i\|^2 \leq a_i^2 2^n \omega_n \epsilon^n$, and by Cauchy-Schwarz

$$|\langle f_{i,\epsilon}, f_{j,\epsilon}\rangle| \leq \delta_{ij} + C_k \epsilon^{n/2},$$

where $C_k$ depends on $a_1, .., a_k$.

**The set $\{df_{i,\epsilon}\}_{i=1}^k$ is almost orthogonal.** Let us obverse that

$$df_{i,\epsilon} = d\chi_\epsilon f_i + \chi_\epsilon df_i,$$

and

$$\langle df_{i,\epsilon}, df_{j,\epsilon}\rangle = \langle d\chi_\epsilon f_i, d\chi_\epsilon f_j\rangle + \langle d\chi_\epsilon f_i, \chi_\epsilon df_j\rangle + \langle \chi_\epsilon df_i, d\chi_\epsilon f_j\rangle + \langle \chi_\epsilon df_i, \chi_\epsilon df_j\rangle.$$

This implies

$$\|d\chi_\epsilon f_i\|^2 \leq C\frac{1}{\epsilon^2}\int_{B(x,2\epsilon)} f_i^2 \leq C a_i^2 2^n \omega_n \epsilon^{n-2}$$

where $C$ depends only on the derivative $\chi'$ of $\chi$. This is the place where $n \geq 3$ is used in order to have a positive exponent to $\epsilon$.

With the same considerations as before

$$|\langle df_{i,\epsilon}, df_{j,\epsilon}\rangle| \leq |\langle df_i, df_j\rangle + C_i \epsilon^{n/2} = \lambda_i \delta_{ij} + C_k \epsilon^{n/2}.$$

where $C_k$ depends on $a_1, ..., a_k, b_1, ..., b_k, C$.

Now, we can estimate the Rayleigh quotient of a function $f \in [f_{1,\epsilon}, ..., f_{k,\epsilon}]$. This is done exactly in the same spirit as the proof of the monotonicity, but we have to deal with the fact that the basis of test functions is only *almost*-orthonormal.

Let $f = \alpha_1 f_{1,\epsilon} + ... + \alpha_k f_{k,\epsilon}$. Thanks to the above considerations, we have

$$\|f\|^2 = \alpha_1^2 + ... + \alpha_k^2 + O(\epsilon^{n/2}),$$
$$\|df\|^2 = \alpha_1^2\lambda_1 + ... + \alpha_k^2\lambda_k + O(\epsilon^{\frac{n-2}{2}}).$$

This implies

$$R(f) \leq \lambda_k + O(\epsilon^{\frac{n-2}{2}})$$

and gives the result.

The next example explains how to produce arbitrarily small eigenvalues for Riemannian manifold with fixed volume and *bounded diameter*.

**The Cheeger dumbbell construction for Riemannian manifolds.** The idea is to consider two n-spheres of fixed volume $A$ connected by a small cylinder $C$ of length $2L$ and radius $\epsilon$. We call denote by $M_\epsilon$ this manifold. The first nonzero eigenvalue of $M_\epsilon$ converges to 0 as $\epsilon$ goes to 0. It is even possible to estimate very precisely the asymptotic of $\lambda_1$ in term of $\epsilon$ (see [An2]), but here, let us just shows that it converges to 0.

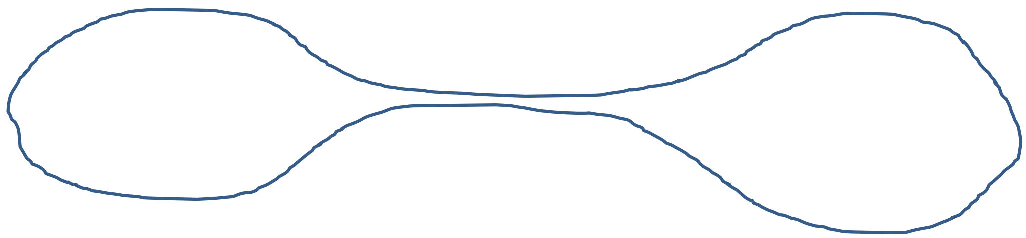
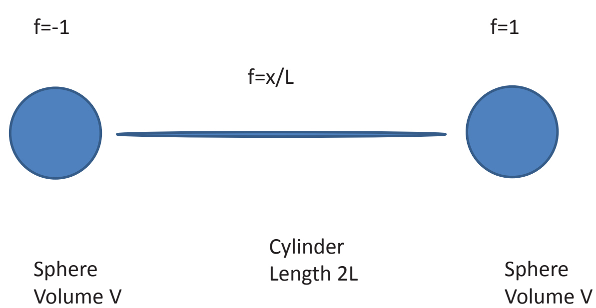


FIGURE 3. Cheeger Dumbell

Let $f$ be a function with value 1 on the first sphere, $-1$ on the second and decreasing linearly along the cylinder.




The norm of its gradient is $\frac{1}{L}$. By construction (and because we can suppose for simplicity that the manifold is symmetric), we have $\int_{M_\epsilon} f dvol_{g_\epsilon} = 0$.

Let $V$ be the vector space generated by $f$ and by the constant function 1. If $h \in V$, we can write $h = a + bf$, $a, b \in \mathbb{R}$ and

$$\int_{M_\epsilon} h^2 dvol_{g_\epsilon} = a^2 Vol(M_\epsilon) + b^2 \int_{M_\epsilon} f^2 dvol_{g_\epsilon},$$

$$\int_{M_\epsilon} |dh|^2 dvol_{g_\epsilon} = b^2 \int_{M_\epsilon} |df|^2 dvol_{g_\epsilon}.$$

So, as $\lambda_1 = inf\{R(h) : h \in V\}$, we get $\lambda_1(M_\epsilon) \leq R(f)$.

The function $f$ varies only on the cylinder $C$ and its gradient has norm $\frac{1}{L}$. This implies

$$\int_{M_\epsilon} |df|^2 dvol_{g_\epsilon} = \frac{1}{L^2} Vol(C).$$

Moreover, because $f^2$ takes the value 1 on both spheres of volume $A$, we have

$$\int_{M_\epsilon} f^2 dvol_{g_\epsilon} \geq 2A.$$

This implies that the Rayleigh quotient of $f$ is bounded above by

$$\frac{VolC/L^2}{2A}$$

which goes to 0 as $\epsilon$ does.

A similar construction with $k$ spheres connected by thin cylinders shows that there exist examples with $k$ arbitrarily small eigenvalues.

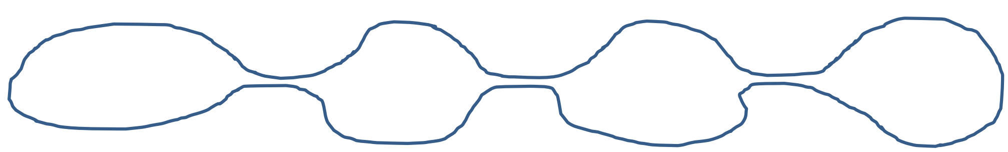


FIGURE 4. Examples with 3 "small" nonzero eigenvalues

Observe that we can easily fix the volume and the diameter in all these constructions: Thus to fix the volume and diameter is not enough to have a lower bound on the spectrum.

**Cheeger dumbbell for the Neumann problem.** The same construction as above may be done in the case of the Neumann problem in $\mathbb{R}^n$. A Cheeger dumbbell is simply given by two disjointed balls related by a thin full cylinder. This gives two interesting examples. The first shows that we have no monotonicity for the Neumann problem.

**Example 18.** *Let $\Omega \subset \mathbb{R}^n$, a domain with a smooth boundary. There exists a domain $\Omega_1 \subset \Omega$ such that the k first eigenvalues of $\Omega_1$ are arbitrarily small.*

*To do this, choose $\Omega_1$ to be a like a Cheeger dumbbell, with $k+1$ balls related by very thin cylinders. We see immediately that the k first eigenvalues can be made as small as we wish, with the same calculations as above.*

*Note that we have no monotonicity for the Neumann problem even when we restrict ourselfto* convex *domains. For example, in dimension 2, we can consider a square $[0, \pi] \times [0, \pi]$. The first nonzero eigenvalue is 1. But we can construct inside this square a thin rectangle around the diagonal of the square of length almost $\sqrt{2}\pi$: the first eigenvalue is close to $\frac{1}{2}$.*

*Note, however, that we cannot find* convex *subsets of $[0, \pi] \times [0, \pi]$ with arbitrarily small first nonzero eigenvalue. By a result of Payne and Weinberger [PW], we have for $\Omega \subset \mathbb{R}^n$, convex, the inequality*

$$\lambda_1(\Omega) \geq \frac{\pi^2}{diam(\Omega)^2}.$$

**Example 19.** *This example shows that a small, local perturbation of a domain may strongly affect the spectrum of the Laplacian for the Neumann conditions. The text gives only qualitative arguements, the calculations are left to the reader if he needs them to understand the examples.*

*We begin with a domain $\Omega$ (a ball in the picture) and show that a very small perturbation of the boundary may change drastically the spectrum for the Neumann boundary condition: A nonzero eigenvalue arbitrarily close to zero may appear as a consequence of a very small perturbation.*

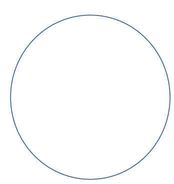


FIGURE 5. We want to perturb locally a domain $\Omega$

*We add a small ball close to $\Omega$ (but at a positive distance) and link it to $\Omega$ using a thin cylinder. The important point is not the size of the ball or of the cylinder, but the ratio. As on the picture, the cylinder has to be much thinner than the ball is small. This creates a Cheeger dumbbell: the only difference with the above calculations is that the two "thick" parts do not have the same size, but it is easy to adapt the calculations and show that if the radius of the cylinder goes to zero, then the same is true for the first nonzero eigenvalue. The ball can be very small and very close to $\Omega$, so the perturbation may be confined in an arbitrarily small region.*

*We can iterate this first deformation, constructing a new deformation on the first, a new one on the second, etc.*

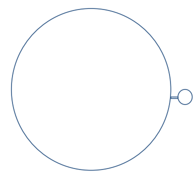


FIGURE 6. Add a small ball and link it to $\Omega$ with a thin cylinder

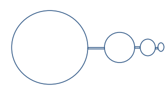


FIGURE 7. Add a small ball and link it to $\Omega$ with a thin cylinder

*This can be understood as a family of balls $B_k$ of radius $r_k = \frac{1}{2^{2k}}$, with center $x_k$. All the centers are on the same line, and the distance between $x_k$ and $x_{k+1}$ is $\frac{1}{2^k}$.*

*The balls $B_k$ and $B_{k+1}$ are joined using a cylinder of radius $\frac{1}{2^{2^k}}$. This radius is very thin in comparison with the radius of the ball.*

*The result of this construction is a domain with an infinite number of Cheeger dumbbells. The domain is not compact, but taking its adherence add only one point, which is of course a singular point. In conclusion, we get a domain with a smooth boundary up to one singular point, and infinitely many small eigenvalues: This implies that the spectrum is not discrete and shows that we must be careful with the smoothness of the boundary.*

**Conclusion:** More generally, by a *Cheeger dumbbell*, we understand a manifold where two (or more)"thick parts" are separated by one (or more) "thin parts".

In such situations, the apparition of small eigenvalue(s) may be suspected.

**A very general question** is to understand under which conditions we can get lower bounds for the spectrum, and in particular for the first nonzero eigenvalue $\lambda_1$.

## 3. Examples of some classical results given lower bound for the first nonzero eigenvalue

As mentioned in the preamble, the second part of this lecture will be concern with the subject *upper bounds for the spectrum.* One of the participants to the school asked me if it is more important or difficult to find upper bounds then lower bounds. The answer is no. The question about *lower bounds* is a world by itself, and the goal of this short section is to present *the tip of the iceberg*: Two different types of results are briefly explained.

In some sense, there are two ways to bound from below the first nonzero eigenvalue of a compact Riemannian manifold. After understanding which deformations of the Riemannian metric can produce small eigenvalues, we have the choice between two strategies.

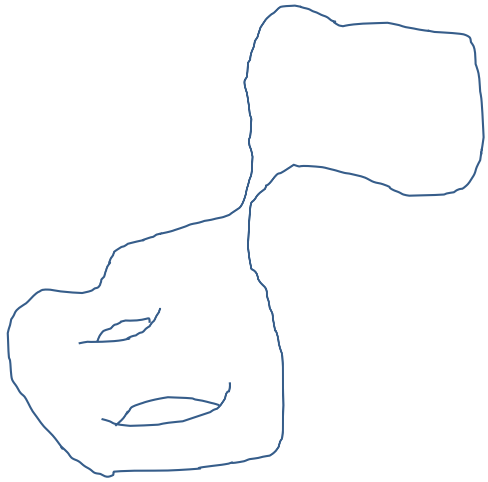


FIGURE 8. "Generalized" Cheeger dumbbell

- A comparison with a geometric constant *taking account* of such local or global deformations of the Riemannian metric or

- To impose geometric constraints *in order to avoid* such local or global deformations of the Riemannian metric.

It has already been explained which deformations of the Riemannian metric can produce small eigenvalues:

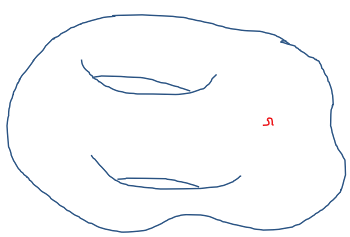


FIGURE 9. It can be because of the local geometry

It can be local deformations, typically a deformation of the metric around a point producing a dumbbell (one says also sometimes that we construct a small *mushroom* on the manifold). It can be a global deformation, adding a *long nose* to the manifold.

Of course, it is not clear that these types of deformations are the only ones producing small eigenvalues, and the difficulty is to find constraints which allow us to control the spectrum from below.

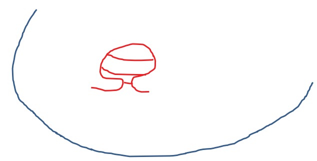


FIGURE 10. “Small mushroom” or a local Cheeger dumbbell

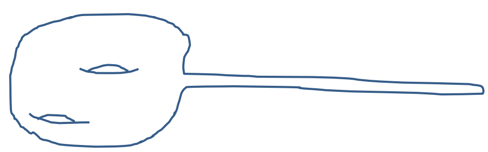


FIGURE 11. It can be because of the global geometry

3.1. **A first method: Cheeger's inequality.** The Cheeger inequality is in some sense the counter-part of the dumbbell example. It is presented here in the case of a compact Riemannian manifold without boundary, but it may be generalized to compact manifolds with boundary (for both Neumann or Dirichlet boundary conditions) or to non-compact, complete, Riemannian manifolds.

**Definition 20.** *Let $(M, g)$ be an $n$-dimensional compact Riemannian manifold without boundary. The Cheeger's isoperimetric constant $h = h(M)$ is*

$$h(M) = \inf_C \{J(C);\ J(C) = \frac{Vol_{n-1}C}{min(Vol_n M_1, Vol_n M_2)}\},$$

*where $C$ runs through all compact codimension one submanifolds which divide $M$ into two disjoint connected open submanifolds $M_1$, $M_2$ with common boundary $C = \partial M_1 = \partial M_2$.*

**Theorem 21.** *(*Cheeger's inequality, 1978*). We have the inequality*

$$\lambda_1(M,g) \geq \frac{h^2(M,g)}{4}.$$

A proof may be found in Chavel's book [Ch1] and developments and other statement in Buser's paper [Bu1]. In particular, Buser shows that Cheeger's inequality is sharp ([Bu1], thm. 1.19).

This inequality is remarkable, because it relates an analytic quantity ($\lambda_1$) to a geometric quantity ($h$) without any other assumption on the geometry of the manifold. Note that the Cheeger constant becomes small in the presence of a Cheeger dumbbell or of a long cylinder. The Cheeger constant takes account of the deformations producing small eigenvalues. Note that it is not always easy to estimate the Cheeger constant $h$ of a Riemannian manifold. Sometimes, it is easier to estimate the first nonzero eigenvalue $\lambda_1$.

Under some geometrical assumptions there is an *upper bound* of $\lambda_1$ in terms of the Cheeger constant (see [Bu2]).

**Theorem 22.** *(Buser, 1982) Let $(M^n,g)$ be a compact Riemannian manifold with Ricci curvature bounded below $Ric(M,g) \geq -\delta^2(n-1)$, $\delta \geq 0$. Then we have*

$$\lambda_1(M,g) \leq C(\delta h + h^2),$$

*where $C$ is a constant depending only on the dimension and $h$ is the Cheeger constant.*

The condition about the Ricci curvature is necessary. In [Bu3], Buser constructs a surface with an arbitrarily small Cheeger constant, but with $\lambda_1$ uniformly bounded from below. It is easy to generalize it to any dimension. This example is very interesting, and we will describe the original construction of Buser in detail.

**Example 23.** *Let $S^1 \times S^1$ be a torus with its product metric $g$ and coordinates $(x,y)$, $-\pi \leq x,y \leq \pi$ and a conformal metric $g_\epsilon = \chi_\epsilon^2 g$.*

*The function $\chi_\epsilon$ is an even function depending only on $x$, takes the value $\epsilon$ at $0,\pi$, 1 outside an $\epsilon$-neighborhood of $0$ and $\pi$.*

*It follows immediately that the Cheeger constant $h(g_\epsilon) \to 0$ as $\epsilon \to 0$ and it remains to show that $\lambda_1(g_\epsilon)$ is uniformly bounded from below.*

*Note that the example seems to be very close to the Cheeger dumbbell. The difference is that in the case of the dumbbell, the length of the cylinder joining the two "big" parts may be small but is fixed. Here, the length of the cylinder goes to $0$ as $\epsilon \to 0$.*

*Let $f$ be an eigenfunction for $\lambda_1(g_\epsilon)$. We have*

$$\lambda_1(g_\epsilon) = R(f) = \frac{\int_T |df|_\epsilon^2 dvol_{g_\epsilon}}{\int_T f^2 dvol_{g_\epsilon}}.$$

*Let $S_1 = \{p : f(p) \geq 0\}$ and $S_2 = \{p : f(p) \leq 0\}$ and let*

$$F(p) = \begin{cases} f(p) & \text{if } p \in S_1 \\ af(p) & \text{if } p \in S_2 \end{cases}$$

*where $a$ is chosen such that $\int_T F dvol_g = 0$.*

*Let $V$ be the vector space generated by the constant function* 1 *and the function $F$ of integral* 0*. The situation is exactly the same as for the calculation around the Cheeger dumbbell and it follows $\lambda_1(g) \le R_g(F)$.*

*So, it is enough to show that $R_g(F) \le \lambda_1(g_\epsilon)$ :*

$$R_g(F) = \frac{\int_{S_1} |df|_g^2 dvol_g + a^2 \int_{S_2} |df|_g^2 dvol_g}{\int_{S_1} f^2 dvol_g + a^2 \int_{S_2} f^2 dvol_g}.$$

*By the conformal invariance in dimension* 2 *(see Prop. 29),*

$$\int_{S_i} |df|_{g_\epsilon}^2 dV_{g_\epsilon} dvol_{g_\epsilon} = \int_{S_i} |df|_g^2 dvol_g,$$

*and by construction*

$$\int_{S_i} f^2 dvol_{g_\epsilon} \le \int_{S_i} f^2 dvol_g.$$

*This implies*

$$\lambda_1(g_\epsilon) = R_{g_\epsilon}(f) \ge R_g(F) \ge \lambda_1(g).$$

The geometrical idea behind this construction is the following: the geometry changes very quickly. This allows to get a small Cheeger constant, but the first eigenfunction has not enough place to vary, and the first eigenvalue remains essentially unchanged.

3.2. **A second method: control of the curvature and of the diameter.** An upper bound on the diameter is clearly a necessary condition to bound the spectrum from below.

A lower bound on the (Ricci) curvature allows avoiding local deformation such as "mushrooms". More precisely, the Ricci curvature allows us to control *the growth of the volume of the balls:* This is the Theorem of Bishop-Gromov that will explained in section 4, in particular Theorem 34.

Here is a lower bound obtained by Li and Yau in 1979:

**Theorem 24.** *(See [LY]). Let $(M,g)$ be a compact $n$-dimensional Riemannian manifold without boundary. Suppose that the Ricci curvature satisfies $Ric(M,g) \ge (n-1)K$ and that $d$ denotes the diameter of $(M,g)$.*

*Then, if $K < 0$,*

$$\lambda_1(M,g) \ge \frac{exp - (1 + (1 - 4(n-1)^2 d^2 K)^{1/2})}{2(n-1)^2 d^2},$$

*and if $K = 0$, then*

$$\lambda_1(M,g) \ge \frac{\pi^2}{4d^2}.$$

This first inequality is difficult to read. It means essentially that, for large $d$,

$$\lambda_1(M,g) \ge C(n) \frac{e^{-(2(n-1)d\sqrt{K})}}{d^2}.$$

This type of results was generalized in different directions, see for example [BBG].

Note that a control of the diameter and of the curvature also allows us to get *upper bounds* for the spectrum.

**Theorem 25.** *(Cheng Comparison Theorem, 1975,[Che]) Let $(M^n, g)$ be a compact $n$-dimensional Riemannian manifold without boundary. Suppose that the Ricci curvature satisfies $Ric(M, g) \geq (n-1)K$ and that $d$ denotes the diameter of $(M, g)$.*

*Then*

$$\lambda_k(M, g) \leq \frac{(n-1)^2 K^2}{4} + \frac{C(n)k}{d^2}$$

*where $C(n)$ is a constant depending only on the dimension.*

## 4. Estimates on the conformal class

This section is the more technical part of this text. I will present with some details the proof given by Korevaar of the existence of upper bounds for all the eigenvalues in a given conformal class of Riemannian metric. This is based on a text by Grigor'yan, Netrusov and Yau [GNY] that I find particularly enlightening.

### 4.1. Introduction.

Let us begin by recalling the following result from [CD] already mentioned in the first part:

**Theorem 26.** *Let $M$ be any compact manifold of dimension $n \geq 3$ and $\lambda > 0$. Then there exists a Riemannian metric $g$ on $M$ with $Vol(M, g) = 1$ and $\lambda_1(M, g) \geq \lambda$.*

However, if we stay on the *conformal class* of a given Riemannian metric $g_0$, then, we get upper bounds for the spectrum on volume one metrics, and it is the goal of this section to explain this.

**Definition 27.** *If $g_0$ is a Riemannian metric on $M$, a metric $g$ is said to be* conformal to the metric $g_0$ *if there is a differentiable function $f$ on $M$, $f > 0$, such that $g(p) = f^2(p)g_0(p)$ for all $p \in M$.*

*We denote by $[g_0]$ the conformal class of $g_0$, that is the family of all Riemannian metrics $g$ conformal to $g_0$*

**Remark 28.** *Geometrically, if two metrics are conformal, the angle between two tangent vectors is preserved. However, it is difficult to see "intuitively" that two Riemannian metrics (for example two submanifolds) are conformal.*

For the proof, only one specific property is needed:

**Proposition 29.** *Let $M$ be a manifold of dimension $n$ and $g$ a Riemannian metric. Let $h$ be a differentiable function on $M$. Then, the expression*

$$|\nabla^g h(p)|_g^n dvol_g$$

*is a conformal invariant, where $dvol_g$ denotes the volume form. This means that, if $g(p) = f^2(p)g_0(p)$, then, for each $p \in M$, we have*

$$|\nabla^g h(p)|_g^n dvol_g = |\nabla^{g_0} h(p)|_{g_0}^n dvol_{g_0}$$

**Proof.** This is a purely linear algebra arguement. If $dimM = n$ and if $g(p) = f^2(p)g_0(p)$, then $dvol_g(p) = f^n(p)dvol_{g_0}$;

If $h$ is a function, $|\nabla^g h(p)|_g^2 = \frac{1}{f^2(p)}|\nabla^{g_0} h(p)|_{g_0}^2$.

This implies that $|\nabla^g h(p)|_g^n dvol_g = |\nabla^{g_0} h(p)|_{g_0}^n dvol_{g_0}$, and the expression

$$|\nabla^g h(p)|_g^n dvol_g$$

is a conformal invariant.

**Remark 30.** (1) *If a metric $g_0$ is conformally deformed, even keeping the volume equal to 1, we have neither control of the diameter nor of the curvature.*
(2) *It is easy to produce arbitrarily small eigenvalues on a conformal class keeping the volume fixed: the Cheeger dumbbell type construction may be done via a conformal deformation of the metric around a point (see [CE1]).*
(3) *For a more complete story of the question about "upper bounds on the conformal class", one can read the introduction of [CE1].*

Our goal is to prove the following theorem from Korevaar [Ko]:

**Theorem 31.** *Let $(M^n, g_0)$ be a compact Riemannian manifold. Then, there exists a constant $C(g_0)$ depending on $g_0$ such that for any Riemannian metric $g \in [g_0]$*

$$\lambda_k(M,g)Vol(M,g)^{2/n} \leq C(g_0)k^{2/n}.$$

*Moreover, if the Ricci curvature of $g_0$ is non-negative, we can replace the constant $C(g_0)$ by a constant depending only on the dimension $n$.*

In the special case of surfaces, the bound depends only on the topology.

**Theorem 32.** *Let $S$ be an oriented surface of genus $\gamma$. Then, there exists a universal constant $C$ such that for any Riemannian metric $g$ on $S$*

$$\lambda_k(S,g)Vol(S) \leq C(\gamma+1)k.$$

The approach of Korevaar for the proof of this theorem is very original and not easy to understand. Around 10 years after the publication of [Ko], Grigor'yan, Netrusov and Yau wrote a paper [GNY] where they explained in particular the main ideas of [Ko] from a more general and more conceptual viewpoint. This paper was used recently in a lot of different situations, in particular in order to find upper bounds for the spectrum of the Steklov problem [CEG1]. In this lecture, I will explain how to use one of the main results of [GNY].

**Remark 33.** (1) *Recall that $\lambda_k(M,g)Vol(M,g)^{2/n}$ is invariant through homothety of the metric, and this control is equivalent to fixing the volume.*
(2) *The estimate is compatible with the Weyl law:*

$$\lambda_k(M,g)Vol(M,g)^{2/n} \leq C(g_0)k^{2/n} \text{ implies } \frac{\lambda_k(M,g)Vol(M,g)^{2/n}}{k^{2/n}} \leq C(g_0).$$

*But this leads to a natural question: Can we get a similar estimate depending asymptotically only on the dimension and on $k$, but not on $g_0$ or on the genus?*

*This is part of the Ph.D. thesis of A. Hassannezhad (see Theorem 41 and Corollary 43 in this paper).*

(3) *These estimates are not sharp in general.*

(4) *These results were already known for $k = 1$, with different kinds of proofs and different authors (see for example the introduction of [CE1]). However, in order to make a proof for all $k$, Korevaar used a completely new approach.*

4.2. **The Bishop-Gromov estimate.** Before going into the description of the proof, let us recall a classical result of Riemannian geometry which appears very often in such situation: the theorem of Bishop-Gromov.

The Bishop-Gromov inequality allows us to control the *growth of balls*. Let us begin with 3 simple examples:

The volume of a ball of radius $r$ in $\mathbb{R}^n$ is $c_n r^n$ and we have

$$\frac{Vol(B(2r))}{Vol(B(r))} = 2^n.$$

The volume of a ball of radius $r$ in the hyperbolic space $\mathbb{H}^n$ is of the type $c_n e^{(n-1)r}$ (for $r$ large enough) and we have

$$\frac{Vol(B(2r))}{Vol(B(r))} \sim e^{(n-1)r},$$

and the ratio grows to $\infty$ as $r \to \infty$.

In the hyperbolic space with curvature $-a^2$, the volume of a ball of radius $r$ is of the type $c_n e^{a(n-1)r}$ and, in this case

$$\frac{Vol(B(2r))}{Vol(B(r))} \sim e^{a(n-1)r}.$$

The ratio growths to $\infty$ for a *fixed radius* as $a \to \infty$.

More generally, on a given manifold $(M, g)$, what can be said about the ratio $\frac{Vol(B(2r))}{Vol(B(r))}$? How is it possible to control it, even for manifold with "non-simple" topology? The result of Bishop-Gromov says that the situations we have described in the hyperbolic spaces are the worst case.

**Theorem 34.** *(Bishop-Gromov, see [Sa] for a proof). Let $(M, g)$ be a complete Riemannian manifold (without boundary): if $Ricci(M, g) \geq -(n-1)a^2 g$, with $a \geq 0$, then for $x \in M$ and $0 < r < R$,*

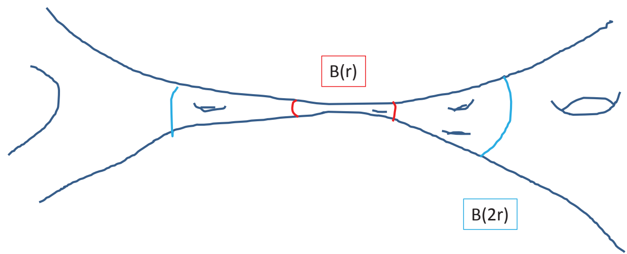
$$\frac{VolB(x,R)}{VolB(x,r)} \leq \frac{VolB^a(x,R)}{VolB^a(x,r)}$$

*where $B^a$ denotes the ball on the model space of constant curvature $-a^2$.*

If $a = 0$, that is if $Ricci(M, g) \geq 0$, the ratio $\frac{VolB(x_i,2r)}{VolB(x_i,r)}$ is controlled by a similar ratio as in the Euclidean space, and this depends only on the dimension!

If $a > 0$, the control of the ratio $\frac{VolB(x_i,2r)}{VolB(x_i,r)}$ is exponential in $r$.

However, if $M$ is compact, this ratio is controlled in function of the diameter of $(M, g)$ and of $a$.




Another way to read this inequality is that

$$Vol(B(x,2R)) \leq \frac{VolB^a(x,R)}{VolB^a(x,r)} Vol(B(x,R)).$$

This means that the volume of a ball of radius $2R$ is controlled by the volume of a ball of radius $R$. If the manifold is compact, this control depends only on $a$ and on the diameter.

In particular, if $(M,g)$ is compact, it is easy to see, using the Bishop-Gromov inequality, that the number of disjoint balls of radius $r/2$ contained in a ball of radius $r$ is controlled by the lower bound $a$ of the Ricci curvature and the diameter of $(M,g)$ (see [Zu] Lemma 3.6, p.230.)

**Definition 35.** *This number (maximal number of disjoint balls of radius $R$ contained in a ball of radius $2R$) is called* the packing constant*.*

**Strategy of the proof of Korevaar's theorem.** During the proof, the way to get upper bounds is to construct test functions, and, as said at point (9) of Theorem 10, it is nice to have disjointly supported functions. These test functions are associated to a *convenient* family of disjoint domains. We will see that the construction of Korevaar ([Ko], [GNY]) gives us a family of *disjoint annuli.* Let us first see how we can associate a test function to an annuli.

4.3. **Test functions associated to an annuli.** This is word-for-word the method of [GNY] as it is described in section 2 of [CES], Lemma 2.1.

Let us fix a reference metric $g_0 \in [g]$ and denote by $d_0$ the distance associated to $g_0$. An *annulus* $A \subset M$ is a subset of $M$ of the form $\{x \in M : r < d_0(x,a) < R\}$ where $a \in M$ and $0 \leq r < R$ (if necessary, we will denote it $A(a,r,R)$). The annulus $2A$ is by definition the annulus $\{x \in M : r/2 < d_0(x,a) < 2R\}$.

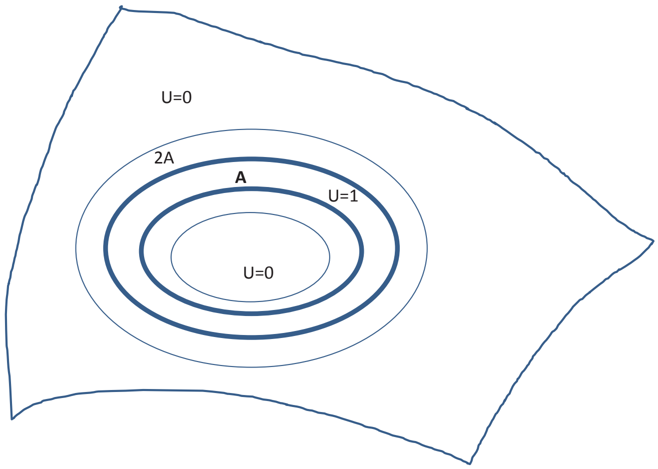



FIGURE 12. Test function associated to an annulus

To such an annulus we associate the function $u_A$ supported in $2A$ and such that

$$u_A(x) = \begin{cases} 1 - \frac{2}{r} d_0(x, A) & \text{if } \frac{r}{2} \le d_0(x, a) \le r \\ 1 & \text{if } x \in A \\ 1 - \frac{1}{R} d_0(x, A) & \text{if } R \le d_0(x, a) \le 2R \end{cases}$$

**Remark 36.** *On a Riemannian manifold $(M, g)$, we use frequently the distance $d$ associated to $g$ in order to construct new functions. It could be, like here, the distance to a subset $A$ of $M$. Note that, even if the boundary of $A$ is not smooth, the function $d(\partial A, \cdot)$ "distance to the boundary of $A$" is well known to be 1-Lipschitz on $M$. According to Rademacher's theorem (see Section 3.1.2, page 81–84 in [EG]), $d(\partial A, \cdot)$ is differentiable almost everywhere (since $dvol_g$ is absolutely continuous with respect to Lebesgue's measure), and its $g$-gradient satisfies $|\nabla d(\partial A, \cdot)|_g \le 1$, almost everywhere.*

We introduce the following constant:

$$\Gamma(g_0) = \sup_{x \in M, r > 0} \frac{V_{g_0}(B(x, r))}{r^n}$$

where $B(x, r)$ stands for the ball of radius $r$ centered at $x$ in $(M, d_0)$. Notice that since $M$ is compact, the constant $\Gamma(g_0)$ is finite and depends only on $g_0$. This constant can be bounded from above in terms of a lower bound of the Ricci curvature $Ric_{g_0}$ and an upper bound of the diameter $diam(M, g_0)$ (Bishop-Gromov inequality). In particular, if the Ricci curvature of $g_0$ is non-negative, then $\Gamma(g_0)$ is bounded above by a constant depending only on the dimension $n$.

**Lemma 37.** *For every annulus $A \subset (M, d_0)$ one has*

$$\int_M |\nabla^g u_A|_g^2 dvol_g \le 8\ \Gamma(g_0)^{\frac{2}{n}} \left( \int_{2A} dvol_g \right)^{1 - \frac{2}{n}} = 8\ \Gamma(g_0)^{\frac{2}{n}} Vol_g(2A)^{1 - \frac{2}{n}}.$$

*Proof.* Let $A = A(a,r,R)$ be an annulus of $(M,d_0)$. Since $u_A$ is supported in $2A$ and we get, using Hölder inequality,

$$\int_M |\nabla^g u_A|_g^2 dvol_g = \int_{2A} |\nabla^g u_A|_g^2 dvol_g \le \left(\int_{2A} |\nabla^g u_A|_g^n dvol_g\right)^{\frac{2}{n}} \left(\int_{2A} dvol_g\right)^{1-\frac{2}{n}}.$$

From the conformal invariance of $\int_{2A} |\nabla^g u_A|_g^n dvol_g$ we have

$$\int_{2A} |\nabla^g u_A|_g^n dvol_g = \int_{2A} |\nabla^{g_0} u_A|_{g_0}^n dvol_{g_0}$$

with

$$|\nabla^{g_0} u_A| \overset{a.e.}{=} \begin{cases} \frac{2}{r} & \text{if } \frac{r}{2} \le d_0(x,a) \le r \\ 0 & \text{if } r \le d_0(x,a) \le R \\ \frac{1}{R} & \text{if } R \le d_0(x,a) \le 2R. \end{cases}$$

Hence,

$$\int_{2A} |\nabla^{g_0} u_A|_{g_0}^n dvol_{g_0} \le \left(\frac{2}{r}\right)^n V_{g_0}(B(a,r)) + \left(\frac{1}{R}\right)^n V_{g_0}(B(a,2R)) \le 2^{n+1}\Gamma(g_0)$$

where the last inequality follows from the definition of $\Gamma(g_0)$. Putting together all the previous inequalities, we obtain the result of the Lemma. □

In order to prove Theorem 31 and 32, we need to consider the Rayleigh quotient of the test-functions. It is given by

$$R(u_A) = \frac{\int_M |\nabla^g u_A|_g^2 dvol_g}{\int_M u_A^2 dvol_g},$$

and we have

$$R(u_A) \le \frac{8\ \Gamma(g_0)^{\frac{2}{n}} Vol_g(2A)^{1-\frac{2}{n}}}{\int_{2A} u_A^2 dvol_g}.$$

So, in order to conclude, we need to control the term $Vol(2A)^{1-\frac{2}{n}}$ from above and the term $\int_{2A} u_A^2 dvol_g$ from below. The first point is easy, but for the second, we need to know about the existence of annuli of *rather large* measure, which is difficult, because we do not have a good geometrical understanding of the geometry under conformal deformations. This is precisely the essential contribution of Korevaar, revisited by Grigor'yan, Netrusov and Yau, to do this.

4.4. **The construction of Korevaar and Grigor'yan-Netrusov-Yau, [Ko], [GNY].** The construction is metric and it is explained in the context of metric measured spaces.

**Definition 38.** *Let $(X,d)$ be a metric space. The* annulus*, denoted by $A(a,r,R)$, (with $a \in X$ and $0 \le r < R$) is the set*

$$A(a,r,R) = \{x \in X : r \le d(x,a) \le R\}.$$

*Moreover, if $\lambda \ge 1$, we will denote by $\lambda A$ the annulus $A(a, \frac{r}{\lambda}, \lambda R)$.*

Let a metric space $(X,d)$ with a finite measure $\nu$. We make the following hypothesis about this space:

(1) The ball are precompact (the closed balls are compact);

(2) The measure $\nu$ is non-atomic;
(3) There exists $N > 0$ such that, for each $r$, a ball of radius $r$ may be covered by at most $N$ balls of radius $r/2$.

This hypothesis plays, in some sense, the role of a control of the curvature, but, as we will see, it is much weaker. Note that it is purely metric, and has nothing to do with the measure, even if, in order to show its validity, we use the Bishop-Gromov estimate about the volume growth of balls.

If these hypotheses are satisfied, we have the following result

**Theorem 39.** *For each positive integer $k$, there exists a family of annuli $\{A_i\}_{i=1}^k$ such that*

(1) *We have $\nu(A_i) \geq C(N)\frac{\nu(X)}{k}$, where $C(N)$ is an explicit constant depending only on $N$;*
(2) *The annuli $2A_i$ are disjoint from each other.*

4.5. **Application. Proof of Theorem 31.** The metric space $X$ will be the manifold $M$ with the Riemannian distance associated to $g_0$ and the *measure* $\nu$ will be the measure associated to the volume form $dvol_g$.

Note that this is *one of the crucial ideas of the proof*: we endowe $M$ with the distance associated to the model Riemannian metric $g_0$, but with a measure associated to the Riemannian metric $g$ to be studied.

As $M$ is compact, the theorem of Bishop-Gromov gives us a constant $C(N) = C(g_0) > 0$, such that, for each $r > 0$ and $x \in M$,

$$\frac{Vol_{g_0} B(x,r)}{Vol_{g_0} B(x,r/2)} \leq C(g_0);$$

The constant $C(g_0)$ depends on the lower bound of $Ricci(g_0)$ and of the diameter of $(M, g_0)$, and only on the dimension if the Ricci curvature is non-negative.

**Remark 40.** *In the sequel, different constants depending on the lower bound of $Ricci(g_0)$ and of the diameter of $(M, g_0)$ will appear: they are denoted by $C(g_0)$. It would be an interesting exercise to make these constants really explicit.*

The goal is to find an upper bound for $\lambda_k(g)$. To this aim, let us take a family of disjoint $2k+2$ annuli given by Theorem 39 and satisfying $Vol_g(A_i) \geq C(g_0)\frac{Vol_g(M)}{k}$ (note that the same constant $C(g_0)$ as in the previous step could be used without lost of generality).

Using for each of the annuli $A$ the test function $u_A$ defined above, we recall that

$$R(u_A) \leq \frac{8\ \Gamma(g_0)^{\frac{2}{n}} Vol(2A)^{1-\frac{2}{n}}}{\int_{2A} u_A^2 dv_g}, \ R(u_A) \leq \frac{\Gamma(g_0)^{\frac{2}{n}} Vol_g(2A)^{1-\frac{2}{n}}}{Vol_g(A)}$$

Moreover, by Theorem 39,

$$Vol_g(A) \geq C(g_0)\frac{Vol_g(M)}{k}.$$

As we have $2k+2$ disjoint annuli, at least $k+1$ of them have a measure less than $\frac{Vol_g(M)}{k}$. So,

$$R(u_A) \leq \Gamma(g_0)^{\frac{2}{n}} \left(\frac{Vol_g(M)}{k}\right)^{1-\frac{2}{n}} \frac{k}{C(g_0)Vol_g(M)} = C(g_0)\left(\frac{k}{Vol_g(M)}\right)^{2/n},$$

and this allows us to prove Theorem (31)

Moreover, if the Ricci curvature of $g_0$ is bounded below by 0, $\Gamma(g_0)$ depends only on the dimension $n$ of $M$ and it is the same for $C(g_0)$.

**Proof of Theorem 32.** Let $(S,g)$ be a compact orientable surface of genus $\gamma$. It is known as a corollary of the Riemann-Roch theorem (see [YY]) that $(S,g)$ admits a conformal branched cover $\psi$ over $(\mathbb{S}^2, g_0)$ with degree $\deg(\psi) \leq \gamma+1$.

Let us endow $\mathbb{S}^2$ with the usual spherical distance $d_0$ associated to the canonical metric $g_0$, and with a measure associated to $g$, the push-forward measure $\nu = \psi_*(dvol_g)$. For any open set $\mathcal{O} \subset \mathbb{S}^2$,

$$\nu(\mathcal{O}) = \int_{\psi^{-1}(\mathcal{O})} dvol_g. \tag{12}$$

We apply Theorem 39 to the metric measure space $(\mathbb{S}^2, d_0, \nu)$: Note that the constant $c = c\,(N(\mathbb{S}^2, d_0))$ arising from Theorem 39 depends now only on the canonical distance on the sphere $S^2$. It can be seen as a universal constant.

We deduce that there exist an absolute constant $c = c\,(N(\mathbb{S}^2, d_0))$ and $k+1$ annuli $A_1, \dots, A_{k+1} \subset \mathbb{S}^2$ such that the annuli $2A_1, \dots, 2A_{k+1}$ are mutually disjoint and, $\forall i \leq k$,

$$\nu(A_i) \geq c\,\frac{\nu(\mathbb{S}^2)}{k}. \tag{13}$$

For each $i \leq k$, set $v_i = u_{A_i} \circ \psi$. From the conformal invariance of the energy and Lemma 37, one deduces that, for every $i \leq k$,

$$\int_S |\nabla_g v_i|_g^2 dvol_g, = \deg(\psi) \int_{\mathbb{S}^2} |\nabla_{g_0} u_{A_i}|_{g_0}^2\, dvol_{g_0} \leq 8\Gamma(\mathbb{S}^2, g_0)(\gamma+1),$$

while, since $u_{A_i}$ is equal to 1 on $A_i$,

$$\int_S v_i^2 dvol_g \geq Vol_g\left(\psi^{-1}(A_i)\right) = \nu(A_i) \geq c\,\frac{\nu(\mathbb{S}^2)}{k} = c\,\frac{Vol_g(S)}{k}.$$

Therefore,

$$R(v_i) \leq \frac{8\Gamma(\mathbb{S}^2, g_0)}{cVol_g(S)}(\gamma+1)k \leq \frac{C(\gamma+1)}{Vol_g(S)}\,k$$

where $C$ is an absolute constant. Noticing that the $k+1$ functions $v_1, \dots v_{k+1}$ are disjointly supported in $S$, we deduce the desired inequality for $\lambda_k(S,g)$.

4.6. **A recent development: the result of A. Hassannezhad.** The upper bounds obtained by N. Korevaar are good in the sense where they are compatible with the Weyl law: Asymptotically, they are of the type $c(g_0)k^{2/n}$ in dimension greater than 2 and $C(genus)k$ in dimension 2. By [CD],[CE1], some dependance on the geometry or in the topology is needed. On the other side, the Weyl law tells us that for a given compact Riemannian manifold of dimension $n$, the k-th eigenvalue is asymptotically of the type $C(n)k^{2/n}$. So, it is natural to ask if one can get uniform upper bounds, asymptotically of the type $C(n)k^{2/n}$. This is the main result of [Ha].

**Theorem 41.** *For each $n \geq 2$, there exist two positive constants $A_n, B_n$such that, for every compact Riemannian manifold $(M, g)$ of dimension $n$*

$$\lambda_k(M, g)Vol_g(M)^{2/n} \leq A_n V([g])^{2/n} + B_n k^{2/n}$$

The only dependence on the metric $g$ is the term $V([g])$ called the min-conformal volume:

**Definition 42.** *Let $(M, g)$ be a compact Riemannian manifold of dimension $n$. The* min-conformal volume *of $(M, g)$ depends only on the conformal class $[g]$ of $g$ and is defined by*

$$V([g]) = inf\{Vol_{g_0}(M) : g_0 \in [g];\ Ric(g_0) \geq -(n-1)\}.$$

*In particular, if there is $g_0 \in [g]$ with $Ric(g_0) \geq 0$, then $V([g]) = 0$.*

Theorem (41) shows that the asymptotic in $k$ is controlled by $B(n)k^{2/n}$ :

$$\frac{\lambda_k(M, g)Vol_g(M)^{2/n}}{k^{2/n}} \leq A(n)\frac{V([g_0])^{2/n}}{k^{2/n}} + B_n,$$

and

$$\frac{V([g_0])^{2/n}}{k^{2/n}} \to 0$$

as $k \to \infty$.

In the case of surfaces, A. Hassannezhad got

**Corollary 43.** *There exist two constants $A, B$ such that for every compact surface $(S, g)$ of genus $\gamma$, we have*

$$\lambda_k(S, g)Vol_g(S) \leq A\gamma + Bk,$$

and, in particular, the asymptotic in $k$ is $Bk$.

**Remark 44.** *Note that Corollary 43 applies for both orientable and non-orientable surfaces.*

For the proof these theorems, the method consists in constructing disjointly supported test functions associated to disjoint subsets of the manifold. The new ingredient is to mix the method of Korevaar with a construction due to Colbois and Maerten [CM]. Roughly speaking, the method of Korevaar gives us annuli, and the estimates become bad when these annuli are too large. The main idea is to use the method of [CM] in such situations.

**Remark 45.** *Each time we have a result of the type*

$$\lambda_k \leq C(geometry)\left(\frac{k}{Vol}\right)^{2/n}$$

*one can ask if one can get a result of the type*

$$\lambda_k Vol^{2/n} \leq A(n)C(geometry) + B(n)k^{2/n}.$$

*Most of the questions of this type are open: An example will be given in section 5, Remark 49.*

## 5. Another geometric method to construct upper bounds and applications.

In this section, I will present (with much less detail) another method to find upper bounds, which was established by D. Maerten and myself [CM], with one application to the spectrum of hypersurfaces. Roughly speaking, the method of [Ko], [GNY] is adequate for conformal metrics, but needs a comparison with one Riemannian metric. In the previous results, all the statements depend on metric $g_0$.

The method of [CM] does not require such a comparison, but it supposes that there is no concentration of the metric. Let us state the version of [CEG]:

**Theorem 46.** *Let $(X, d, \mu)$ be a complete, locally compact metric measured space, where $\mu$ is a finite measure. We assume that for all $r > 0$, there exists an integer $N(r)$ such that each ball of radius $r$ can be covered by $N(r)$ balls of radius $r/2$.*

*If there exist an integer $K > 0$ and a radius $r > 0$ such that, for each $x \in X$*

$$\mu(B(x, r)) \leq \frac{\mu(X)}{4N^2(r)K},$$

*then, there exist $K$ $\mu$-measurable subsets $A_1, ..., A_K$ of $X$ such that, $\forall i \leq K$, $\mu(A_i) \geq \frac{\mu(X)}{2N(r)K}$ and, for $i \neq j$, $d(A_i, A_j) \geq 3r$.*


**Remark 47.** (1) *The first hypothesis is essentially the same as in the construction of Korevaar and suppose a control of the packing constant.*

(2) *In the version given in [CM], there is the hypothesis that the volume of the $r$-balls tends to 0 uniformly on $X$ as $r \to 0$. This is not necessary; it is enough that the measure $\mu(B(x, r))$ of the ball is small enough.*

(3) *In order to get a bound for $\lambda_k$ comparable with the Weyl law, $r$ has to be of the size of $(\frac{C\mu(X)}{k})^{1/n}$, and $K = k+1$.*
(4) *If $r$ is chosen too small, the upper bounds we get are far from being optimal.*

As in the previous construction we associate to each subset $A_i$ the test function $u_{A_i}$ which is 1 inside $A_i$ and 0 outside the $r$-neighborhood $A_i^r$ of $A_i$. If $A$ is one of the subsets $A_i$ and $A^r = \{p \in X : d(p, A) \leq r\}$

$$u_A(x) = \begin{cases} 1 - \frac{1}{r}d(x, A) & \text{if } x \in A^r \\ 1 & \text{if } x \in A \\ 0 & \text{if } d(x, A) \geq r \end{cases}$$

**An application.** Let us describe a situation where this method can be used.

Let $\Sigma \subset \mathbb{R}^{m+1}$ be a compact hypersurface. As a consequence of the result of [CM] and of a classical theorem of Nash-Kuiper, it is possible to deform $\Sigma$ in $\mathbb{R}^{m+1}$ in order to have

$$\lambda_1(\Sigma) Vol(\Sigma)^{2/m}$$

arbitrarily large. See [CDE], paragraph 5, for a detailed explanation.

The upper bounds for $\Sigma$ will depend of the *isoperimetric ratio* of the hypersurface. Let $\Omega \subset \mathbb{R}^{m+1}$ be a bounded domain with smooth boundary $\Sigma = \partial\Omega$. We denote by

$$I(\Omega) = \frac{Vol(\Sigma)}{Vol(\Omega)^{m/(m+1)}}$$

the isoperimetric ratio of $\Omega$.

**Theorem 48.** *(C-El Soufi-Girouard, [CEG2]) For any bounded domain $\Omega \subset \mathbb{R}^{m+1}$ with smooth boundary $\Sigma = \partial\Omega$, and all $k \geq 1$,*

$$\lambda_k(\Sigma)|\Sigma|^{2/m} \leq C_m I(\Omega)^{1+2/m} k^{2/m} \tag{14}$$

*with $C_m$ an explicit constant depending on $m$, and $I(\Omega)$ denotes the isoperimetric ratio.*

The proof, given in [CEG], uses the construction of [CM]: First, we need to introduce $(X, d, \mu)$ be a complete, locally compact metric measure space, where $\mu$ is a finite measure satisfying the first hypothesis: For all $r > 0$, there exists an integer $N(r)$ such that each ball of radius $r$ can be covered by $N(r)$ balls of radius $r/2$. In our situation, $X = \mathbb{R}^{m+1}$, $d$ is the Euclidean distance. This implies that the packing constant $N$ depends only on the dimension $m$. The measure $\mu$ is the measure associated to the hypersurface $\Sigma$:

$$\mu(U) = Vol(U \cap \Sigma).$$

The main difficulty for the proof is that the second hypothesis is not satisfied. It is not always true that there exists an integer $K > 0$ and a radius $r > 0$ such that, for each $x \in X$ $\mu(B(x, r)) \leq \frac{\mu(X)}{4N^2(r)K}$.. The reason is that the measure can concentrate some place.

Recall that for $r_k$ of the type $\left(\frac{C\mu(\Sigma)}{k}\right)^{1/m}$, we hope to have $\mu(B(x, r_k)) \leq \frac{\mu(\Sigma)}{4C(m)k}$.

The *idea* of the proof of Theorem 48 is as follows.

The difficult part of the proof (that I will not explain here) is to show that the hypothesis about the concentration is correct, but only *for a part of* $\Sigma$, proportional to the measure $\mu(\Sigma)$, with factor of proportionality $\frac{C(m)}{I(\Omega)}$.

This leads us to choose

$$r = \left(\frac{C_1(m)\mu(\Sigma)}{kI(\Omega)}\right)^{1/m}.$$

The Rayleigh quotient of a test function $u_A$ can be estimated as follows. If $dv_\Sigma$ denotes the volume form on $\Sigma$,

$$\int_\Sigma |\nabla u_A|^2 dv_\Sigma \leq \frac{1}{r_k^2}\mu(A^{r_k})$$

and

$$\int_\Sigma u_A^2 dv_\Sigma \geq \frac{\mu(\Sigma)}{C_2(n)I(\Omega)k}$$

we get

$$R(u_A) \leq \frac{1}{r_k^2}\frac{\mu(A^{r_k})}{\mu(\Sigma)}C_2(n)I(\Omega)k$$

and we have a problem with le term $\mu(A^{r_k})$... As in the proof of the theorem of Korevaar, we can begin with $2k+2$ subset, so that, for at least $(k+1)$ of them, $\mu(A^{r_k}) \leq \frac{\mu(\Sigma)}{k+1}$. This implies

$$R(u_A) \leq \frac{1}{r_k^2}\frac{\mu(\Sigma)}{k+1}\frac{C_2(n)I(\Omega)k}{\mu(\Sigma)} \leq C(n)I(\Omega)^{1+\frac{2}{m}}\left(\frac{k}{Vol(\Sigma)}\right)^{2/m}.$$

**Remark 49.** *This is a typical situation where the open question occurs: is it possible to establish an inequality of the type*

$$\lambda_k(\Sigma)Vol(\Sigma)^{2/m} \leq A_m I(\Omega)^{1+\frac{2}{m}} + B_m k^{2/m}$$

*for two constant $A_m, B_m$ depending only on the dimension $m$?*

*For this specific question the answer is no (thesis of Luc Pétiard, article in preparation for 2016).*

## 6. The conformal spectrum

Having upper (or lower) bounds leads to investigate the spectrum from a qualitative or quantitative viewpoint. Let us give the example of the *conformal spectrum* and of the *topological spectrum* we developed in [CE1] with A. El Soufi (see also [Co] for a short survey).

**Definition 50.** *For any natural integer $k$ and any conformal class of metrics $[g_0]$ on $M$, we define the conformal $k$-th eigenvalue of $(M,[g_0])$ to be*

$$\lambda_k^c(M,[g_0]) = \sup\left\{\lambda_k(M,g)Vol(M,g)^{2/n} \mid g \text{ is conformal to } g_0\right\}.$$

*The sequence $\{\lambda_k^c(M,[g_0])\}$ constitutes the conformal spectrum of $(M,[g_0])$.*

*In dimension 2, one can also define a topological spectrum by setting, for any genus $\gamma$ and any integer $k \geq 0$,*

$$\lambda_k^{top}(\gamma) = \sup\left\{\lambda_k(M,g)Vol(M,g)\right\},$$

*where $g$ describes the set of Riemannian metrics on the orientable compact surface $M$ of genus $\gamma$.*

Regarding the quantitative aspects, in some very special situations, the hope is to detect the maximum or the supremum in a conformal class. This concerns mainly the first nonzero eigenvalue. Note that for the topological spectrum on surfaces, it was already explained in (1.3.2) that it was a very difficult question. It is as well difficult to decide if the supremum is a maximum: Does it a Riemannian metric $g \in [g_0]$ exists such that $\lambda_k Vol(M,g)^{2/n}$ is maximum?

Regarding the conformal first eigenvalue, El Soufi and Ilias [EI] gave a sufficient condition for a Riemannian metric $g$ to maximize $\lambda_1$ in its conformal class $[g]$: if there exists a family $f_1, f_2, ..., f_p$ of first eigenfunctions satisfying $\sum_i df_i \otimes df_i = g$, then $\lambda_1^c(M,[g]) = \lambda_1(g)$. This condition is fulfilled in particular by the metric of any homogeneous Riemannian space with irreducible isotropy representation. For instance, the first conformal eigenvalues of the rank one symmetric spaces endowed with their standard conformal classes $[g_s]$, are given by

- $\lambda_1^c(\mathbb{S}^n,[g_s]) = n\omega_n^{2/n}$, where $\omega_n$ is the volume of the $n$-dimensional Euclidean sphere of radius one,
- $\lambda_1^c(\mathbb{R}P^n,[g_s]) = 2^{\frac{n-2}{n}}(n+1)\omega_n^{2/n}$,
- $\lambda_1^c(\mathbb{C}P^d,[g_s]) = 4\pi(d+1)d!^{-1/d}$,
- $\lambda_1^c(\mathbb{H}P^d,[g_s]) = 8\pi(d+1)(2d+1)!^{-1/2d}$,
- $\lambda_1^c(\mathbb{C}aP^2,[g_s]) = 48\pi(\frac{6}{11!})^{1/8} = 8\pi\sqrt{6}(\frac{9}{385})^{1/8}$.

**Remark 51.** *Note that this is a strong hypothesis to suppose that the maximum is reached by a* regular *Riemannian metric. This cannot be expected in general, and even in dimension 2, it is a fundamental question to understand what kinds of singularities may occurs for the maximal metrics (with respect to $\lambda_1$ or $\lambda_k$).*

I will mainly describe the qualitative aspects of the problem and the first qualitative question is the following: On a compact manifold $M$, does a Riemannian metric $g$ exists such that $\lambda_1^c[g]$ is arbitrarily small? The answer is no: Among all the possible conformal classes of metrics on manifolds, the standard conformal class of the sphere is the one having the lowest conformal spectrum.

**Theorem 52.** *For any conformal class $[g]$ on $M$ and any integer $k \geq 0$,*

$$\lambda_k^c(M,[g]) \geq \lambda_k^c(\mathbb{S}^n,[g_s]).$$

Although the eigenvalues of a given Riemannian metric may have nontrivial multiplicities, the conformal eigenvalues are all simple: The conformal spectrum consists of a strictly increasing sequence, and, moreover, the gap between two consecutive conformal eigenvalues is uniformly bounded. Precisely, we have the following theorem:

**Theorem 53.** *For any conformal class $[g]$ on $M$ and any integer $k \geq 0$,*

$$\lambda_{k+1}^c(M,[g])^{n/2} - \lambda_k^c(M,[g])^{n/2} \geq \lambda_1^c(\mathbb{S}^n,[g_s]) = n^{n/2}\omega_n,$$

*where $\omega_n$ is the volume of the $n$-dimensional Euclidean sphere of radius one.*

An immediate consequence of these two theorems is the following explicit estimate of $\lambda_k^c(M,[g])$:

**Corollary 54.** *For any conformal class $[g]$ on $M$ and any integer $k \geq 0$,*

$$\lambda_k^c(M,[g]) \geq n\omega_n^{2/n}k^{2/n}.$$

Combined with the Hassannezhad estimates [Ha], Corollary 54 gives

$$n\omega_n^{2/n}k^{2/n} \leq \lambda_k^c(M,[g]) \leq A_n V([g])^{2/n} + B_n k^{2/n}$$

for some constants $A_n, B_n$ depending only on $n$, and $V([g])$ depending on a lower bound of $Ric\, d^2$, where $Ric$ is the Ricci curvature and $d$ is the diameter of $g$ or of another representative of $[g]$). This implies the relation

$$n\omega_n^{2/n} \leq \frac{\lambda_k^c(M,[g])}{k^{2/n}} \leq A_n \left(\frac{V([g])}{k}\right)^{2/n} + B_n.$$

Asymptotically, as $k \to \infty$, this shows that $\frac{\lambda_k^c(M,[g])}{k^{2/n}}$ is between $n\omega_n^{2/n}$ and $B_n$.

**Remark 55.** *It would be interesting to determine more explicitly the value of $B_n$ in Hassannezhad estimates [Ha].*

Corollary 54 implies also that, if the $k$-th eigenvalue $\lambda_k(g)$ of a metric $g$ is less than $n\omega_n^{2/n}k^{2/n}$, then $g$ does not maximize $\lambda_k$ on its conformal class $[g]$. For instance, *the standard metric $g_s$ of $\mathbb{S}^2$, which maximizes $\lambda_1$, does not maximize $\lambda_k$ on $[g_s]$ for any* $k \geq 2$. This fact answers a question of Yau (see [Y], p. 686).

El Soufi and Ilias have also shown that the property for a Riemannian metric to be critical (in a generalized sense) implies multiplicity. As a corollary of this and of the previous results, we get

**Corollary 56.** *If a Riemannian metric $g$ maximizes $\lambda_1$ on its conformal class $[g]$, then it does not maximize $\lambda_2$ on $[g]$. More generally, a Riemannian metric $g$ cannot maximize simultaneously three consecutive eigenvalues $\lambda_k, \lambda_{k+1}, \lambda_{k+2}$ on $[g]$.*

Note however that it is a difficult question to decide if there exists a Riemannian metric $g_1 \in [g]$ such that $\lambda_k^c([g]) = \lambda_k(g_1)$.

Regarding the topological spectrum of surfaces, we have similar results:

**Theorem 57.** *For any fixed genus $\gamma$ and any integer $k \geq 0$, we have*

$$\lambda_{k+1}^{top}(\gamma) - \lambda_k^{top}(\gamma) \geq 8\pi,$$

*and*

$$\lambda_k^{top}(\gamma) \geq 8(k-1)\pi + \lambda_1^{top}(\gamma) \geq 8k\pi$$

However, we have good lower bounds for $\lambda_1^{top}(\gamma)$ (see for example [BM]) so that we can say that,

$$\lambda_1^{top}(\gamma) \geq \frac{4}{5}\pi(\gamma-1),$$

and it follows

$$\lambda_k^{top}(\gamma) \geq \frac{4}{5}\pi(\gamma-1) + 8(k-1)\pi.$$

Combined with Hassannezhad estimates [Ha], we get for an oriented surface of genus $\gamma$

$$\frac{4}{5}\pi(\gamma-1)+8(k-1)\pi \leq \lambda_k^{top}(\gamma) \leq A(\gamma-1)+Bk.$$

for $A, B$ constant.

**Remark 58.** *Again, it would be interesting to have a more explicit estimate of the term $B$ in Hassanhezhad estimates, because, asymptotically, $\frac{\lambda_k^{top}(\gamma)}{k}$ is between $8\pi$ and $B$.*

We can also look at the behavior of $\lambda_k^c(\gamma)$ in terms of the genus $\gamma$.

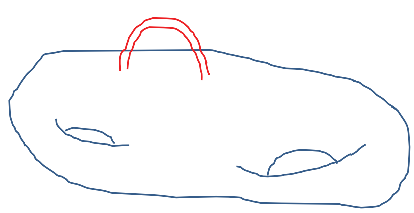
**Theorem 59.** *We have*

$$\lambda_k^{top}(\gamma+1) \geq \lambda_k^{top}(\gamma).$$

Note that this is an *open question* to know if the inequality is strict or not for each $\gamma$.

**A few words about the proof.** Locally, a Riemannian manifold is almost Euclidean and one can deform a small neighbourhood of a point into a (almost) round sphere (blow up). Then, we can use these deformations to get our results, using classical facts about the behavior of the spectrum under surgery. This explains the presence of the first eigenvalue of the round sphere in all these estimates.

In order to show $\lambda_k^{top}(\gamma+1) \geq \lambda_k^{top}(\gamma)$, we have to compare two surfaces with different topology and this needs another idea: to add a thin and short handle in order to pass from a surface of genus $\gamma$ to a surface of genus $\gamma+1$. By classical surgery results, this neither affects the eigenvalues between $\lambda_1$ and $\lambda_k$ (if the handle is short enough) nor the area. However, it cannot be proved with this method that the inequality is strict.



**Remark 60.** *Each time there are lower or upper bounds, it is possible to study the spectrum from a qualitative viewpoint. However, this is not always easy! Conside for example the following question in the case of hypersurfaces. For a positive number $I$ (large enough), define*

$$\lambda_k(I) = sup\{\lambda_k(\Sigma)Vol(\Sigma)^{2/m},\}$$

*where $\Sigma = \partial\Omega$, $\Omega$ domain in $\mathbb{R}^{m+1}$ such that $I(\Omega) \leq I$.*

*What can be said about $\{\lambda_k(I)\}$?*

*For example, do we have $\lambda_{k+1}(I) > \lambda_k(I)$? What about the gap $\lambda_{k+1}(I) - \lambda_k(I)$?*

**Recent results about this topic.**

(1) For the second conformal eigenvalue of the canonical metric on the sphere, asymptotically sharp upper bound were found in [GNP] and [Pe1].

(2) In [Pe2], Petrides proved the following result: let $(M, g)$ be a compact Riemannian manifold of dimension $n \geq 3$. The first conformal eigenvalue is always greater than $n\omega_n^{2/n}$ (the value it takes for the round sphere) except if $(M, g)$ is conformally diffeomorphic to the standard sphere.

(3) In the specific case of the first eigenvalue for compact surfaces, there have been several works recently published about the existence and the regularity of maximal metrics: Roughly speaking, the maximal metrics are not expected to be regular, but singular with very few singularities consisting only in conical points [Kok], [Pe3].

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